

Holographic Wilsonian flows and emergent fermions in extremal charged black holes

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Abstract

We study holographic Wilsonian RG in a general class of asymptotically AdS backgrounds with a $U(1)$ gauge field. We consider free charged Dirac fermions in such a background, and integrate them up to an intermediate radial distance, yielding an equivalent low energy dual field theory. The new ingredient, compared to scalars, involves a ‘generalized’ basis of coherent states which labels a particular half of the fermion components as coordinates or momenta, depending on the choice of quantization (standard or alternative). We apply this technology to explicitly compute RG flows of charged fermionic operators and their composites (double trace operators) in field theories dual to (a) pure AdS and (b) extremal charged black hole geometries. The flow diagrams and fixed points are determined explicitly. In the case of the extremal black hole, the RG flows connect two fixed points at the UV AdS boundary to two fixed points at the IR AdS_2 region. The double trace flow is shown, both numerically and analytically, to develop a pole singularity in the AdS_2 region at low frequency and near the Fermi momentum, which can be traced to the appearance of massless fermion modes on the low energy cut-off surface. The low energy field theory action we derive exactly agrees with the semi-holographic action proposed by Faulkner and Polchinski in [19]. In terms of field theory, the holographic version of Wilsonian RG leads to a quantum theory with random sources. In the extremal black hole background the random sources become ‘light’ in the AdS_2 region near the Fermi surface and emerge as new dynamical degrees of freedom.

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1 Introduction and Summary

Recently, a holographic description of field theories with a finite cut-off has been proposed in [1–4]. One of the imports of this proposal is a holographic version of Wilsonian renormalization group (hWRG), in which non-trivial RG flows of strongly interacting field theories are described with consummate ease in terms of free fields in the bulk. The basic idea of hWRG is to integrate out the bulk field from the boundary up to some intermediate radial distance, yielding an equivalent dual field theory at a lower cut-off. This procedure has been exactly implemented for a free scalar field in a fixed background geometry [1, 2]¹; see [6–10] for further related developments, as well as, *e.g.*, [11–16] for earlier treatments of holographic renormalization group (the relationship between the earlier treatments and hWRG has been

¹ [1, 2] also discuss gauge field fluctuations. See [5] for a related discussion on metric fluctuations, although from a somewhat different viewpoint.

discussed in [1]). Although it is not easy to identify the specific dual field theories which arise in this fashion,² it is of interest to explore the general structure of RG flows and fixed points in these theories. Indeed, this program, with the inclusion of metric fluctuations and interactions, can provide an RG landscape of cut-off field theories and their universality classes holographically: something which is of obvious importance in applied AdS/CFT.

The main purpose of this note is to extend this method to fermions (see also [18]). A technical motivation is that, because the Dirac action is of first order, the fermion components contain both coordinates and momenta; thus, it is not *a priori* clear how to define the ‘Dirichlet boundary condition’ for fermions (see Section 2 for comparison with scalars). As we will see below, the definition needs the construction of some suitably ‘generalized’ coherent states. A more physical motivation for considering the fermion problem is to understand the origin of the semi-holographic theory of [19] which gives a simple understanding of the models of [20, 21] near the Fermi surface³. In [19], it is argued that the UV theory flows to an IR theory which, near the Fermi surface, is described by a dual AdS_2 geometry plus a ‘domain wall’ fermion. We will discover both the relevance of the Fermi surface and the origin of this additional fermion in hWRG: (a) the Fermi surface can be precisely characterized by a singular behaviour of the double trace coupling as the hWRG flow is continued to the near horizon AdS_2 region, (b) the result of hWRG can be interpreted as a field theory with a lower cut-off with a random source: when the cut-off surface is in the AdS_2 region the random source becomes massless and becomes an emergent degree of freedom in the low energy theory, which is to be identified with the ‘domain wall’ fermion of [19].

We describe below the plan of the paper and the basic results:

Section 2 gives a review of hWRG for scalars, including some new material which include explicit RG flow diagrams in pure AdS and in extremal charged black hole (ECBH) backgrounds. The flow diagrams show, respectively, two and four fixed points in these two backgrounds (see Figure 1). The field theory interpretation of these findings is described in Section 7.

In Section 3 we consider free fermions in a general class of asymptotically AdS backgrounds with a $U(1)$ gauge field, which are translationally invariant in the ‘boundary’ directions. As in the case of the bosonic hWRG, the idea is to integrate out the bulk fermions from the boundary down to an intermediate radial distance, yielding an equivalent low energy dual field theory. A crucial difference from the bosonic hWRG is the need to use a basis of states in the bulk Hilbert space which are eigenstates of one half of the fermion components (the ‘coordinates’); we construct such a basis by using ‘generalized’ coherent states which distinguish the ‘appropriate half’ in question.⁴ The hWRG gives rise to flows involving products of pairs of single-trace fermionic operators; in contrast with the bosonic example, there are different double trace operators in the fermionic case. We write down the general flow equations for these operators in the general background and a set of simplified flow equations in rotationally symmetric backgrounds.

In Section 4, we provide an exact solution of the hWRG equations in pure AdS backgrounds (with a constant electrostatic potential). As in the scalar case, double trace flows interpolate between two fixed points (with 0 and 1 relevant directions), which, inside the Klebanov-Witten window [24], correspond to two different choices of quantization: standard and alternative (see Figure 3). More details of the field theory interpretation are provided in Section 7.

In Section 5, we compute RG flows in an ECBH background. The ECBH background has two AdS regions: AdS_{d+1} at the boundary and $\text{AdS}_2 \times \mathbb{R}^{d-1}$ in the near-horizon region. Developing further on the pure AdS results of the previous section, we first compute RG flows in the two AdS regions. We then complete these flows in the full background numerically. The RG flows feature four fixed points, two each in the two AdS regions. The two fixed points in the asymptotically AdS_{d+1} region both become unstable in the ECBH background (acquiring, respectively, 2 and 1 relevant directions) and flow down to the two fixed points in the interior AdS_2 region (which continue to have 0 and 1 relevant directions). We parametrize the flow down from AdS_{d+1} to AdS_2 by the flow of a geometric quantity, namely the red-shift factor (as in Section 2); near the AdS_{d+1} boundary, this can be regarded in the dual field theory as the flow of the stress tensor T_{tt} , as described further in Section 7. There are parameter ranges in which the two fixed

²See Section 7 for a brief qualitative comparison with a matrix field theory with a Wilson-Fisher fixed point; details will appear in [17].

³See [2] for an elucidation of the semi-holographic bosonic model of [4].

⁴Similar states were proposed and constructed in [22, 23].

points in the AdS_2 region merge and annihilate, which is reminiscent of [25].

In Section 6, we use the formalism developed above to give a first-principles derivation of the semi-holographic action of Faulkner and Polchinski [19] for the extremal black hole. We show, numerically and analytically, that the double trace coupling, in the process of hWRG flow, develops a pole singularity at zero frequency and Fermi momentum on a cut-off surface in the AdS_2 region, leading to a non-local dual field theory. We trace the origin of the non-locality to integration over a fermion present in the theory, which becomes massless on the cut-off surface at the Fermi momentum. The low energy theory, therefore, can be written in a local form, by combining this emergent fermion with the operators of the dual field theory, precisely as in [19]. Remarks to this effect were briefly mentioned earlier in [2] (see also [26]). Effects of double trace deformations to AdS/CMT have been considered in [27, 28].

In Section 7 we briefly summarize the various results of the previous sections in field theoretic terms. In particular, we briefly mention a comparison of the bosonic hWRG with a matrix field theory in $4 - \epsilon$ dimensions with a Wilson-Fisher fixed point. We also briefly mention an interpretation of hWRG in terms of a random source, and interpret the emergence of a massless fermionic degree of freedom in terms of a fluctuating fermionic source. We include a comparison of the random source fields with mesons in the Gross-Neveu model [29] and in the Nambu–Jona-Lasinio approach to QCD [30, 31].

In Section 8 we conclude with a few open questions. The appendices describe various technical details, including a summary of the construction of the ‘generalized’ coherent states.

Work described in this paper has been presented at various stages of development in various meetings [32]. Fermionic Wilsonian flows in pure AdS have been discussed in [18] using somewhat different methods.

2 Holographic Wilsonian RG: scalars

This section will contain a review of some of the results in [1, 2] and a brief account of some new computations (details will appear in [17]). We will consider [1, 2] a free bulk scalar $\phi(z, x^\mu)$ ⁵ in a fixed asymptotically AdS geometry

$$ds^2 = g_{MN} dx^M dx^N = g_{zz}(z) dz^2 + g_{\mu\nu}(z) dx^\mu dx^\nu. \quad (1)$$

ϕ is dual to a single-trace operator $O(x^\mu)$. Wilsonian RG is implemented by (a) considering the gravity partition function (effectively a partition function over ϕ) as a path integral with a radial Hamiltonian, from a UV cut-off $z = \epsilon_0$ to an IR cut-off $z = \epsilon_{\text{IR}}$, and (b) performing the integral over a slice $z \in [\epsilon_0, \epsilon]$:

$$Z_{\text{grav}}(\epsilon_0) := \langle \text{IR} | \hat{U}(\epsilon_{\text{IR}}, \epsilon_0) | \text{UV} \rangle = \int D\tilde{\phi} \langle \text{IR} | \hat{U}(\epsilon_{\text{IR}}, \epsilon) | \tilde{\phi} \rangle \langle \tilde{\phi} | \hat{U}(\epsilon, \epsilon_0) | \text{UV} \rangle = \int D\tilde{\phi} Z_{\text{grav}}(\epsilon, \tilde{\phi}) e^{S_{\text{uv}}(\tilde{\phi}, \epsilon, \epsilon_0)}. \quad (2)$$

Here $\hat{U}(\epsilon_1, \epsilon_2) := P e^{-\int_{\epsilon_1}^{\epsilon_2} \hat{H}}$ (P denotes radial time ordering) and we will use the parametrization

$$e^{S_{\text{uv}}} \equiv \langle \tilde{\phi} | \hat{U}(\epsilon, \epsilon_0) | \text{UV} \rangle := \exp \left[\int_k \left(-\frac{1}{2} f(k, \epsilon) \tilde{\phi}(k) \tilde{\phi}(-k) + J(k, \epsilon) \tilde{\phi}(-k) \right) + C(\epsilon) \right], \quad (3)$$

where $\int_k := \frac{1}{\kappa^2} \int \frac{d^d k}{(2\pi)^d}$. The wave-functions $|\text{IR}\rangle$ and $|\text{UV}\rangle$ implement the IR and UV boundary conditions. For example, if $|\text{UV}\rangle = |\phi_0\rangle$ we have a Dirichlet b.c. at UV. In that case, the holographic dual to (2) is obtained by replacing Z_{grav} s in (2) with field theory partition functions with sources (assuming that the usual GKP-Witten relation [33, 34] holds at finite cut-offs)

$$\begin{aligned} \left\langle \exp \int_k \phi_0 O \right\rangle_{\epsilon_0}^{\text{std}} &= \int D\tilde{\phi} \left\langle \exp \int_k \tilde{\phi} O \right\rangle_{\epsilon}^{\text{std}} e^{S_{\text{uv}}(\tilde{\phi}, \epsilon, \epsilon_0)} \\ &= \left\langle \exp \left(\int_k \frac{1}{2} g(k, \epsilon) O^2 + \int_k h(k, \epsilon) O + C'(\epsilon) \right) \right\rangle_{\epsilon}^{\text{std}}. \end{aligned} \quad (4)$$

⁵ Described by an action $S(\phi) = -\frac{1}{2} \int d^d x dz \sqrt{g} [\partial_M \phi \partial^M \phi + m^2 \phi^2]$

In the last step, we have performed the $\tilde{\phi}$ integration, leading to a double trace coupling g and a single trace coupling h , where $g(k) = 1/f(k)$, $h(k) = J(k)/f(k)$. Equation (4) relates the field theory at the UV cut-off ϵ_0 to that at the intermediate cut-off ϵ (for small ϵ_0, ϵ we can roughly imagine $\epsilon_0 \sim \Lambda_0^{-1}$, $\epsilon \sim \Lambda^{-1}$, $\Lambda < \Lambda_0$ [35, 36]).⁶ The symbol $\langle\langle \dots \rangle\rangle$ denotes *unnormalized* correlators which we do not divide by the partition function; the subscript ϵ or ϵ_0 denotes the field theory cut-off; the superscript ^{std} denotes the fact that a Dirichlet b.c. in the bulk corresponds to *standard* quantization in the field theory (a Neumann boundary condition, in an appropriate window, corresponds to the so-called *alternative* quantization; see below (8) for more details; see footnote 6 for more general choices of boundary conditions).

Clearly $e^{S_{\text{uv}}}$ satisfies a Schrödinger equation, which, in the semi-classical limit, becomes the Hamilton-Jacobi equation $\partial_\epsilon S = -H(\partial S/\partial \tilde{\phi}, \tilde{\phi})$. In terms of the parametrization (3), the Schrödinger (or Hamilton-Jacobi) equation becomes

$$\frac{\sqrt{g^{zz}}}{\sqrt{\gamma}} \partial_\epsilon (\sqrt{\gamma} f(k, \epsilon)) = -f^2(k, \epsilon) + k^\mu k_\mu + m^2, \quad (5)$$

$$\frac{\sqrt{g^{zz}}}{\sqrt{\gamma}} \partial_\epsilon (\sqrt{\gamma} J(k, \epsilon)) = -J(k, \epsilon) f(k, \epsilon), \quad \frac{\sqrt{g^{zz}}}{\sqrt{\gamma}} \partial_\epsilon C(\epsilon) = \frac{1}{2} \int_k J(k, \epsilon) J(-k, \epsilon), \quad (6)$$

where γ is the determinant of the induced metric at the cut-off surface. In view of the comments below (4) the above equations are analogous to beta-function equations; we will elaborate more on this in Section 7.

AdS_{d+1} In this case, (5) becomes (using the notation $\dot{(\cdot)} := \epsilon \partial_\epsilon(\cdot)$)

$$\dot{f} = -f^2 + d f + \epsilon^2 k^2 + m^2. \quad (7)$$

Now since ϵ is a cut-off scale (see comments below (4)), the explicit appearance of a cut-off in the above equation appears to spoil an immediate interpretation as a beta-function. To remedy this, we define dimensionless couplings \bar{f} by $f(\epsilon, k) := \bar{f}(\epsilon, \bar{k})$ where $\bar{k} := \epsilon k$ is the dimensionless momentum. Making a formal Taylor expansion $\bar{f} = f_0(\epsilon) + f_1(\epsilon)(\bar{k})^2 + \dots$ we get the following coupled flow equations

$$\begin{aligned} \dot{f}_0 &= -f_0^2 + d f_0 + m^2 = -(f_0 - \Delta_+)(f_0 - \Delta_-), \quad \dot{f}_1 = (d - 2 - 2f_0)f_1 + 1, \quad \dots \\ \Delta_\pm &\equiv \frac{d}{2} \pm \nu, \quad \nu = \sqrt{\left(\frac{d}{2}\right)^2 + m^2}. \end{aligned} \quad (8)$$

There are clearly two fixed points given by $f_0 = \Delta_\pm$ (all higher modes f_n are uniquely determined in terms of f_0). Clearly $f_0 = \Delta_+$ is an attractive, hence IR, fixed point of the double trace flow (consider $\delta f := f - \Delta_+$), while $f_0 = \Delta_-$ is a repulsive, hence UV, fixed point. These correspond, respectively, to the standard and alternative quantizations; the field theory interpretations of these statements are detailed in Section 7, and in Table 1.

	dimension of $O(x)$	dimension of $O(x)^2$	nature of $O(x)^2$	nature of fixed point
‘standard’ CFT	Δ_+	$2\Delta_+ = d + 2\nu > d$	irrelevant	IR (attractive)
‘alternative’ CFT	Δ_-	$2\Delta_- = d - 2\nu < d$	relevant	UV (repulsive)

Table 1: Dimensions of operators at the two fixed points. The dimensions can be read off either by a computation of two-point functions, or by using the fact that for an interaction $\int g \mathbb{O}$ with beta-function $\beta := -\dot{g}$, the dimension of the operator \mathbb{O} satisfies $\gamma = \partial \beta / \partial g$ (see Section 7 for more details).

The RG flow in the f_0 - f_1 plane (for $d = 3, m^2 = -2$)⁷ is shown in the left panel of Figure 1.

⁶ Double trace deformations can also be introduced in the bare theory, by choosing the wave-function $|\text{UV}\rangle = \int D\phi_0 \Psi(\phi_0) |\phi_0\rangle$, with $\Psi(\phi_0)$ a Gaussian, like in the RHS of (3). For other formulations of multi-trace deformations in the scalar case, see, for example, [37–40, 2].

⁷For $d = 3$, unitarity requires $m^2 \geq -9/4$ (the Breitenlohner-Freedman (BF) bound) whereas the Klebanov-Witten window is $-5/4 \geq m^2 \geq -9/4$.

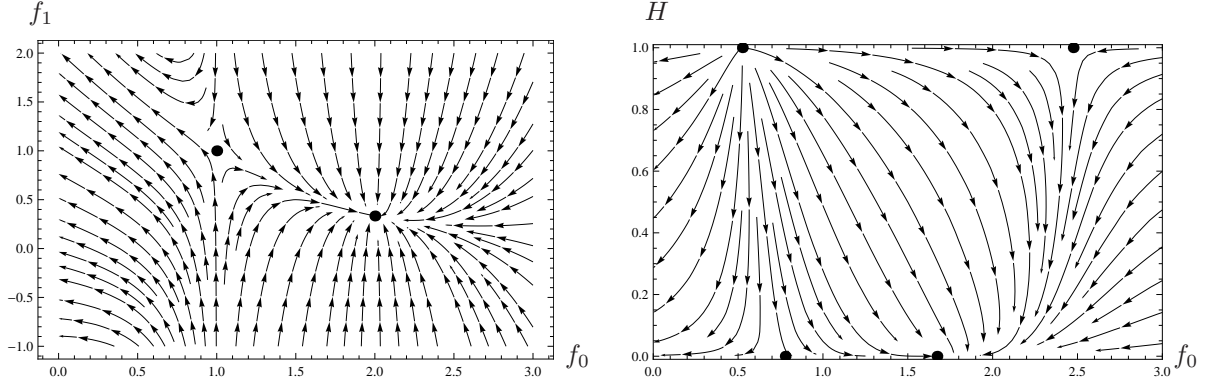


Figure 1: RG flow for double trace couplings: (a) the left panel shows the case of a scalar field in pure AdS background, with double trace flows (8) connecting two fixed points; (b) the right panel shows the case of the extremal BH background, where double trace flows at the top (AdS₄) boundary and at the bottom (AdS₂) boundary are connected by a geometric flow, that of the red-shift factor H .

Extremal Charged BH background In this case, the metric and the gauge potential are given by (using units where the AdS₄ radius is equal to 1)

$$ds^2 = \frac{1}{z^2} (H(z) dt^2 + dx_i dx_i) + \frac{dz^2}{z^2 H(z)}, \quad A = \mu \left(1 - \frac{z}{z_*} \right) dt, \\ H(z) = 1 + \frac{d}{d-2} \left(\frac{z}{z_*} \right)^4 - \frac{2(d-1)}{d-2} \left(\frac{z}{z_*} \right)^3, \quad (9)$$

where z_* is the black hole horizon and $i = 1, \dots, d-1$. The flow equation (5) for f becomes (writing $k_\mu = (\omega, k_i)$)

$$\epsilon \partial_\epsilon (f \sqrt{H}) = d f \sqrt{H} - f^2 + \epsilon^2 (k_i k_i + \omega^2 / H) + m^2. \quad (10)$$

There are now two explicit sources of cut-off dependence on the RHS. We have learned above how to handle the ϵ^2 -term, by introducing a dimensionless coupling \bar{f} . *How about the explicit dependence of H on ϵ ?* The answer is that $H(\epsilon)$ simply reflects the presence of a new Wilsonian-‘type’⁸ coupling in the black hole background, whose ‘beta-function’ is independent of the double trace flow.⁹ Using these facts, and considering the translationally invariant mode of $f(k, \epsilon)$, which is $f_0(\epsilon)$, we get an RG flow in the f_0 - H plane as in the right panel of Figure 1. The plot (computed for $d = 3, m^2 = -1.3$) clearly shows four fixed points: two of them at the AdS₄ boundary ($H = 1$), with $f_0 = \Delta_\pm$; and the remaining two at the AdS₂ boundary ($H = 0$), with $f_0 = \delta_\pm = (1/2 \pm \nu_2)/R_2$, $\nu_2 = \sqrt{1/4 + m^2 R_2^2}$, $R_2^2 = 1/6$; where R_2 is the value of the effective radius of curvature of the emergent AdS₂ in the limit $z \rightarrow z_*$. See Section 7 for a field theory interpretation of this flow diagram.

3 Holographic Wilsonian flow equations for Dirac fermions

Summary of this section In this section, we will consider a free charged Dirac fermion ψ propagating in an asymptotically AdS _{$d+1$} spacetime (12). The holographic dual of this system will be understood to be a large N field theory with a global $U(1)$ current, with a certain charged fermionic single-trace operator O which couples to the bulk fermion ψ . The precise nature of the coupling is described in equations (22) and (23), (24). It will be assumed that the geometry

⁸The ‘quotes’ reflect the fact that $\delta H(z) \sim \delta g_{tt}$ is actually a normalizable deformation, so β_H is to be identified with a flow of the VEV of a stress tensor, rather than with the running of a coupling; see Section 7 for details.

⁹ $\beta_H := \epsilon \partial_\epsilon H$, which is a function of ϵ , can be recast as a function of H by inverting $H(\epsilon)$; $\beta_H(H)$ satisfies the implicit equation

$$(1 - H + \beta_H/3)^3 = (1 - H + \beta_H/4)^4. \quad (11)$$

is not affected by the bulk fermion. We will describe how to integrate out the bulk fermion up to an intermediate radial cut-off $z = \epsilon$. As described in the Introduction and in Section 2, this procedure implements a Wilsonian RG in the dual field theory, and induces a flow of double trace operators (see (33) and (34)) whose beta-function is computed in (37) and (39).

Dirac action Consider a free Dirac fermion, of charge q , in an asymptotically AdS background, with a metric and a $U(1)$ gauge field of the form

$$ds^2 = g_{MN} dx^M dx^N = g_{zz}(z) dz^2 + g_{\mu\nu}(z) dx^\mu dx^\nu, \quad A_z = 0, \quad A_\mu = A_\mu(z). \quad (12)$$

The Dirac action is given by¹⁰

$$S = \int dz d^d x \sqrt{g} (\bar{\psi} \Gamma^M \mathcal{D}_M \psi - m \bar{\psi} \psi). \quad (13)$$

For simplicity, we work with Euclidean signature.¹¹ The kinetic term in the z -direction of the Dirac action can be made into a canonical one,

$$S = \int dz d^d x \left(\bar{\Phi} \Gamma^{\hat{z}} \partial_z \Phi + \sqrt{g_{zz}} [\bar{\Phi} (\Gamma^\mu \partial_\mu - iq \Gamma^\mu A_\mu) \Phi - m \bar{\Phi} \Phi] \right), \quad (14)$$

after field rescalings $\Phi := (g g^{zz})^{1/4} \psi$, $\bar{\Phi} := (g g^{zz})^{1/4} \bar{\psi}$.

In order to implement the strategy indicated in (2), we need to compute the radial Hamiltonian and a complete set of states. A basic difference from the scalar case is that the components of ψ contain both ‘coordinates’ and ‘momenta’. Thus, we cannot have states analogous to field-eigenstates $|\phi\rangle$; rather, it turns out that we need to introduce some kind of coherent states. We will first describe the radial Hamiltonian and the coherent states.

Radial Hamiltonian We adopt the following representations of $(d+1)$ -dimensional Dirac matrices,

$$d = \text{even} \quad \Gamma^{\hat{z}} := \gamma^{d+1}, \quad \Gamma^{\hat{\mu}} := \gamma^{\hat{\mu}}, \quad \chi_\pm := \frac{1 \pm \Gamma^{\hat{z}}}{2} \Phi, \quad \bar{\chi}_\pm := \bar{\Phi} \frac{1 \pm \Gamma^{\hat{z}}}{2}, \quad (15)$$

where γ^{d+1} is the chirality matrix in d -dimension.

$$d = \text{odd} \quad \Gamma^{\hat{z}} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma^{\hat{\mu}} := \begin{pmatrix} 0 & \gamma^{\hat{\mu}} \\ \gamma^{\hat{\mu}} & 0 \end{pmatrix}, \quad \Phi := \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}, \quad \bar{\Phi} := (\bar{\chi}_+, \bar{\chi}_-), \quad (16)$$

In these conventions, the anticommutation relations and the Hamiltonian are given by

$$\hat{H} = \int \frac{d^d k}{(2\pi)^d} \sqrt{g_{zz}} \left[i \hat{\chi}_+ \gamma^\mu K_\mu \hat{\chi}_- + i \hat{\chi}_- \gamma^\mu K_\mu \hat{\chi}_+ + m \hat{\chi}_+ \hat{\chi}_+ - m \hat{\chi}_- \hat{\chi}_- \right], \quad (17)$$

$$\{\hat{\chi}_\pm(z, k_\mu), \pm \hat{\chi}_\pm(z, k'_\mu)\} = (2\pi)^d \kappa^2 \delta^d(k - k'), \quad (18)$$

where we have moved into the momentum space and $K_\mu := k_\mu - q A_\mu$. We can prove that the above radial Hamiltonian and anticommutation relations *indeed* reproduce the classical equations of motion through $\kappa^2 \partial_z \hat{O}(z) = [\hat{H}, \hat{O}(z)]$, as well as the path integral (21), thus constituting a proof of the equations (17) and (18) (see Appendix D).

Boundary conditions and generalized coherent states Let us review boundary conditions imposed on Dirac fields at the boundary $z \sim 0$. Since the Dirac action is of first order, the wavefunction is a function of only half of the components of ψ and $\bar{\psi}$. Accordingly, boundary conditions are imposed on only half of the components. In the case $m > 1/2$, the components ψ_- and $\bar{\psi}_+$ become non-normalizable near the boundary (they behave as $\sim z^{d/2-m}$, see

¹⁰ See Appendix A for definitions of the covariant derivatives and so on.

¹¹ For a general form of the Dirac action which is applicable to both Lorentzian and Euclidean metrics, see Appendix A.2.

(141) and (143)), and thus these components are set to be the boundary sources. The states at the boundary, analogous to $|\text{UV}\rangle$ in (2), are chosen to be eigenstates of these components, defined by¹²

$$\widehat{\chi}_+|\overline{\chi}_+, \chi_-\rangle = \overline{\chi}_+|\overline{\chi}_+, \chi_-\rangle, \quad \widehat{\chi}_-|\overline{\chi}_+, \chi_-\rangle = \chi_-|\overline{\chi}_+, \chi_-\rangle. \quad (19)$$

We will call such states generalized coherent states (see Appendix B). The above choice of boundary states turns out to correspond to standard quantization, as shown shortly.

For $0 < m < 1/2$,¹³ it is possible to choose the other components (namely, $\chi_+, \overline{\chi}_-$) as the boundary sources; the corresponding boundary states are the complementary set

$$\widehat{\chi}_+|\chi_+, \overline{\chi}_-\rangle = \chi_+|\chi_+, \overline{\chi}_-\rangle, \quad \widehat{\chi}_-|\chi_+, \overline{\chi}_-\rangle = \overline{\chi}_-|\chi_+, \overline{\chi}_-\rangle. \quad (20)$$

This choice turns out to correspond to the alternative quantization.

Path integral Based on the generalized coherent states, we can show that

$$\langle \chi_+^1, \overline{\chi}_-^1 | \widehat{U}(\epsilon_1, \epsilon_2) | \overline{\chi}_+^2, \chi_-^2 \rangle = \int [d^4\chi] e^{-\kappa^{-2} S[\chi, \overline{\chi}] + S_{\text{bdy}}}, \quad S_{\text{bdy}} := - \int_k (\overline{\chi}_-^1 \chi_-^1 + \overline{\chi}_+^2 \chi_+^2). \quad (21)$$

where $\chi_-^1 := \chi_-(z = \epsilon_1)$ and $\chi_+^2 := \chi_+(z = \epsilon_2)$, and $S[\chi, \overline{\chi}]$ is given by (14). The proof is given in Appendix D. Since the derivation is independent of details of the radial Hamiltonian, this result applies both to pure AdS_{d+1} case and to a general background case. Thus, we can see that the above choice of the boundary state (19) correctly gives the boundary terms for the standard quantization, which was proposed and justified in a semi-classical manner in [41–43].

The AdS/CFT dictionary As mentioned in the beginning of this section, the bulk fermion ψ is coupled to a certain single trace fermionic operator O in the field theory, which is charged under a global $U(1)$ current. We can now state the precise couplings of the various spinorial components using the χ -notation for the bulk fermions. These are as follows:

$$\chi_- \longleftrightarrow \overline{O}_-, \quad \overline{\chi}_+ \longleftrightarrow O_+; \quad \chi_+ \longleftrightarrow \overline{O}_+, \quad \overline{\chi}_- \longleftrightarrow O_-. \quad (22)$$

In the standard quantization, AdS/CFT duality reads as [41]

$$\left\langle\left\langle e^{\int_k (\overline{\chi}_+ O_+ + \overline{O}_- \chi_-)} \right\rangle\right\rangle_\epsilon^{\text{std}} = \langle \text{IR} | \widehat{U}(\epsilon_{\text{IR}}, \epsilon) | \overline{\chi}_+, \chi_- \rangle, \quad (23)$$

where, as in (4), we have used $\langle\langle \dots \rangle\rangle$ to denote an unnormalized partition function. The subscript ϵ denotes the cut-off at which the field theory is defined (here we are identifying the radial cut-off with the field theory cut-off, see comments below (4)). Note that the initial state is chosen as eigenstates of $\chi_-, \overline{\chi}_+$, in keeping with the boundary condition appropriate for the standard quantization, as indicated by the superscript std .

In the case of the alternative quantization we must choose an initial state which is the eigenstate of the remaining half of the fermion components, giving rise to

$$\left\langle\left\langle e^{\int_k (\overline{O}_+ \chi_+ + \overline{\chi}_- O_-)} \right\rangle\right\rangle_\epsilon^{\text{alt}} = \langle \text{IR} | \widehat{U}(\epsilon_{\text{IR}}, \epsilon) | \chi_+, \overline{\chi}_- \rangle. \quad (24)$$

This dictionary matches with the usual definition of the alternative quantization as a Legendre transformation of the standard quantization, which can be seen from (129) and (131). Both (23) and (24) are ‘bare’ AdS/CFT relations; to define continuum field theories, we need to consider counterterms, as discussed in Section E.2.

¹² A notation rule is that the first (second) column in the state vector corresponds to the $+$ ($-$) chirality.

¹³ This range of masses defines the Klebanov-Witten window for fermions, where both standard and alternative quantizations are allowed.

Multi-trace deformations Following the scalar case (see footnote 6), let us consider a transition amplitude with a general initial state $\langle \text{IR} | \hat{U}(\epsilon_{\text{IR}}, \epsilon) | \text{UV} \rangle$ where $|\text{UV}\rangle$ is described by a wavefunction

$$\langle \chi_+, \bar{\chi}_- | \text{UV} \rangle =: \exp(W[\chi_+, \bar{\chi}_-]) . \quad (25)$$

The AdS/CFT relation in this case, written in standard quantization, can be shown to be¹⁴

$$\langle \text{IR} | \hat{U}(\epsilon_{\text{IR}}, \epsilon) | \text{UV} \rangle = \left\langle\!\!\left\langle e^{W[-O_+, -\bar{O}_-]} \right\rangle\!\!\right\rangle_{\epsilon}^{\text{std}} . \quad (26)$$

To obtain a similar result in terms of alternative quantization, we can expand the wavefunction in the complementary basis by inserting (130) or (131):

$$\langle \bar{\chi}_+, \chi_- | \text{UV} \rangle = \int D\eta_+ D\bar{\eta}_- \exp \left\{ \int_k (\bar{\chi}_+ \eta_+ + \bar{\eta}_- \chi_-) + W[\eta_+, \bar{\eta}_-] \right\} =: \exp(\widetilde{W}[\bar{\chi}_+, \chi_-]) , \quad (27)$$

and find the following representation for the transition amplitude

$$\langle \text{IR} | \hat{U}(\epsilon_{\text{IR}}, \epsilon) | \text{UV} \rangle = \left\langle\!\!\left\langle e^{\widetilde{W}[\bar{O}_+, O_-]} \right\rangle\!\!\right\rangle_{\epsilon}^{\text{alt}} \quad (28)$$

These results mean that the general initial state gives rise to multi-trace deformations of the standard and the alternative quantizations.¹⁵ Thus, for example, if we choose $W[\chi_+, \bar{\chi}_-] = \int_k \bar{\chi}_- a \chi_+$,

$$\langle \text{IR} | \hat{U}(\epsilon_{\text{IR}}, \epsilon) | \text{UV} \rangle = \left\langle\!\!\left\langle e^{\int_k \bar{O}_- a O_+} \right\rangle\!\!\right\rangle_{\epsilon}^{\text{std}} = \left\langle\!\!\left\langle e^{-\int_k \bar{O}_+ a^{-1} O_-} \right\rangle\!\!\right\rangle_{\epsilon}^{\text{alt}} . \quad (29)$$

3.1 Schrödinger equation and RG flow equations

Let us consider a field theory with a bare cut-off ϵ_0 , defined holographically in the standard quantization, through equations such as (23) or (26) with $\epsilon = \epsilon_0$. As in (2) for the scalar case, we slice the bulk path integral into two parts at $z = \epsilon$:

$$\langle \text{IR} | \hat{U}(\epsilon_{\text{IR}}, \epsilon_0) | \text{UV} \rangle = \int D^4\chi \langle \text{IR} | \hat{U}(\epsilon_{\text{IR}}, \epsilon) | \bar{\chi}_+, \chi_- \rangle e^{\int_k (\bar{\chi}_+ \chi_+ + \bar{\chi}_- \chi_-)} \langle \chi_+, \bar{\chi}_- | \hat{U}(\epsilon, \epsilon_0) | \text{UV} \rangle , \quad (30)$$

where we used a completeness relation (130). Using (23), the IR amplitudes $\langle \text{IR} | \dots \rangle$ on both sides of the above equation can be replaced by field theory quantities at cut-offs ϵ_0 and ϵ respectively. Making this replacement on the RHS explicit, we get

$$\langle \text{IR} | \hat{U}(\epsilon_{\text{IR}}, \epsilon_0) | \text{UV} \rangle = \int D^4\chi \left\langle\!\!\left\langle e^{\int_k (\bar{\chi}_+ O_+ + \bar{O}_- \chi_-)} \right\rangle\!\!\right\rangle_{\epsilon}^{\text{std}} e^{\int_k (\bar{\chi}_+ \chi_+ + \bar{\chi}_- \chi_-)} \langle \chi_+, \bar{\chi}_- | \hat{U}(\epsilon, \epsilon_0) | \text{UV} \rangle . \quad (31)$$

Thus the UV amplitude $\langle \dots | \text{UV} \rangle$ on the RHS connects a field theory at a lower cut-off ϵ with that at a higher cut-off ϵ_0 . Since we are considering here a quadratic Hamiltonian, the UV amplitude can be written as an exponential of a quadratic polynomial in $\chi_+, \bar{\chi}_-$,¹⁶

$$\langle \chi_+, \bar{\chi}_- | \hat{U}(\epsilon, \epsilon_0) | \text{UV} \rangle \equiv \Psi[\chi_+, \bar{\chi}_-] := \exp \left\{ - \int_k [\bar{\chi}_- F(\epsilon) \chi_+ + \bar{\chi}_- J_-(\epsilon) + \bar{J}_+(\epsilon) \chi_+ + C(\epsilon)] \right\} , \quad (32)$$

Then, applying the technique used in deriving (26) to the right hand side of (31), we get an important relation

$$\langle \text{IR} | \hat{U}(\epsilon_{\text{IR}}, \epsilon_0) | \text{UV} \rangle = \left\langle\!\!\left\langle \exp \int_k [-\bar{O}_- F(\epsilon) O_+ + \bar{O}_- J_-(\epsilon) + \bar{J}_+(\epsilon) O_+ - C(\epsilon)] \right\rangle\!\!\right\rangle_{\epsilon}^{\text{std}} . \quad (33)$$

¹⁴To derive this, we expand $|\text{UV}\rangle$ in the $|\bar{\chi}_+, \chi_- \rangle$ basis, make a temporary replacement $W[\chi_+, \bar{\chi}_-] \rightarrow W[\chi_+, \bar{\chi}_-] - \int (\bar{J}_+ \chi_+ + \bar{\chi}_- J_-)$, and bring W outside the $\int D^4\chi$ integral as $W(-\partial/\partial \bar{J}_+, \partial/\partial J_-)$. The $\int D^4\chi$ integral now converts $|\bar{\chi}_+, \chi_- \rangle \rightarrow |\bar{J}_+, J_- \rangle$, so that using (23), the W -term becomes $W[-O_+, -\bar{O}_-]$. At this stage, we can put $J = 0$ and arrive at (26).

¹⁵ For other formulations of multi-trace deformations in the fermion case, see [40].

¹⁶In this and various other equations, the k -dependence of the χ s and of the coefficients F, J are suppressed.

Since the LHS is clearly independent of ϵ , so must be the RHS. This is the statement of RG invariance: the dependence on ϵ of the coefficients $F(\epsilon)$, $J(\epsilon)$ and $C(\epsilon)$ is such that the field theory on the RHS of the above equation, for any choice of the cut-off ϵ , is independent of ϵ , and is equivalent to the original cut-off theory. In terms of alternative quantization, (33) becomes

$$\langle \text{IR} | \hat{U}(\epsilon_{\text{IR}}, \epsilon_0) | \text{UV} \rangle = \left\langle \left\langle \exp \int_k [\bar{O}_+ F(\epsilon)^{-1} O_- - \bar{O}_+ F(\epsilon)^{-1} J_-(\epsilon) - \bar{J}_+(\epsilon) F(\epsilon)^{-1} O_- - C(\epsilon)] \right\rangle \right\rangle_\epsilon^{\text{alt}}. \quad (34)$$

To determine the ϵ -dependence of F , J , C , note that the wavefunction Ψ in (32) satisfies the Schrödinger equation in the radial time ϵ ,

$$-\kappa^2 \partial_\epsilon \langle \eta_+, \bar{\eta}_- | \hat{U}(\epsilon, \epsilon_0) | \text{UV} \rangle = H \left[\eta_+, \bar{\eta}_-, \frac{\delta}{\delta \eta_+}, \frac{\delta}{\delta \bar{\eta}_-} \right] \langle \eta_+, \bar{\eta}_- | \hat{U}(\epsilon, \epsilon_0) | \text{UV} \rangle, \quad (35)$$

where $H \left[\eta_+, \bar{\eta}_-, \frac{\delta}{\delta \eta_+}, \frac{\delta}{\delta \bar{\eta}_-} \right]$ is defined by applying the following replacements to the Hamiltonian \hat{H} in (17),¹⁷

$$\hat{\chi}_+ \longrightarrow \eta_+, \quad \hat{\bar{\chi}}_+ \longrightarrow \kappa_d^2 \frac{\delta}{\delta \eta_+}, \quad \hat{\chi}_- \longrightarrow -\kappa_d^2 \frac{\delta}{\delta \bar{\eta}_-}, \quad \hat{\bar{\chi}}_- \longrightarrow \bar{\eta}_-, \quad \kappa_d^2 := \kappa^2 (2\pi)^d. \quad (36)$$

For the derivation of the replacements, see the end of Appendix B. Furthermore, the same replacement laws can be applied to a general Hamiltonian including interactions, as one can see from the derivation.

It is easy to see that the Schrödinger equation (35) with the Hamiltonian (17) is satisfied if and only if the coefficients (F, \bar{J}_+, J_-, C) in the UV amplitude (32) satisfy the following flow equations,

$$\begin{aligned} \sqrt{g^{zz}} \partial_\epsilon F &= F(i\gamma^\mu K_\mu)F + i\gamma^\mu K_\mu - 2mF, \\ \sqrt{g^{zz}} \partial_\epsilon J_- &= F(i\gamma^\mu K_\mu)J_- - mJ_-, \\ \sqrt{g^{zz}} \partial_\epsilon \bar{J}_+ &= \bar{J}_+(i\gamma^\mu K_\mu)F - m\bar{J}_+, \\ \sqrt{g^{zz}} \partial_\epsilon C &= \bar{J}_+(i\gamma^\mu K_\mu)J_- + \kappa_d^2 \text{tr}(i\gamma^\mu K_\mu F) \delta^d(k=0), \end{aligned} \quad (37)$$

where the trace is taken over spinor indices. Note that the first three equations are classical. One can construct the solutions of the flow equations from classical solutions $\Phi_\pm, \bar{\Phi}_\pm$ of the equations of motion¹⁸ as follows:

$$F = \Phi_- (\Phi_+)^{-1} = (\bar{\Phi}_-)^{-1} \bar{\Phi}_+, \quad J_- = (\bar{\Phi}_-)^{-1} j_-, \quad \bar{J}_+ = \bar{j}_+ (\Phi_+)^{-1}, \quad (38)$$

where j_-, \bar{j}_+ are spinors independent of ϵ .

3.1.1 Rotationally symmetric case

In the case where the background is rotationally symmetric in the $d-1$ spatial boundary directions, the flow equations simplify considerably after using a special choice of the Dirac matrices¹⁹. Details are given in Appendix C.²⁰ If the wave-functional Ψ in (32) is parametrized by (149), then the Schrödinger equation (148) yields the following flow equations

$$\begin{aligned} \sqrt{g^{zz}} \partial_\epsilon F_\alpha &= -F_\alpha T_\alpha^- F_\alpha - 2mF_\alpha + T_\alpha^+, \\ \sqrt{g^{zz}} \partial_\epsilon \bar{J}_\alpha^+ &= -\bar{J}_\alpha^+ T_\alpha^- F_\alpha - m\bar{J}_\alpha^+, \\ \sqrt{g^{zz}} \partial_\epsilon J_\alpha^- &= -F_\alpha T_\alpha^- J_\alpha^- - mJ_\alpha^-, \\ \sqrt{g^{zz}} \partial_\epsilon C_\alpha &= -\bar{J}_\alpha^+ T_\alpha^- J_\alpha^- - \kappa_d^2 T_\alpha^- F_\alpha \delta^d(0), \end{aligned} \quad (39)$$

¹⁷ Note that (neglecting momentum and spin indices) the replacements give a two-dimensional representation of the operator algebra (18) while the Fock space basis gives a four-dimensional representation. The former representation is related to a representation defined in [23], whereas the latter is similar to that defined in [22].

¹⁸ Namely, $\Gamma^\varepsilon \partial_z \Phi + \sqrt{g^{zz}} (\Gamma^\mu \partial_\mu - iq\Gamma^\mu A_\mu) \Phi - m\bar{\Phi} \Phi = 0$.

¹⁹ In this case we use Lorentzian signature.

²⁰ Note that the special choice of the Dirac matrices in Appendix C has different structure from the representation (15), (16). The difference arises from the projection operators Π_α (135).

where

$$T_\alpha^\pm := \pm \sqrt{-g^{tt}} (\omega + qA_t) - (-)^\alpha \sqrt{g^{11}} k. \quad (40)$$

Just like (38), one can construct the solutions to the flow equations from solutions of the classical equations of motion (138). Let $\Phi_\alpha^\pm, \bar{\Phi}_\alpha^\pm$ be the classical solutions of (139). Then, the solutions of the flow equations are given by

$$F_\alpha = \Phi_\alpha^-(\Phi_\alpha^+)^{-1} = (\bar{\Phi}_\alpha^-)^{-1} \bar{\Phi}_\alpha^+, \quad J_\alpha^- = (\bar{\Phi}_\alpha^-)^{-1} j_\alpha^-, \quad \bar{J}_\alpha^+ = \bar{J}_\alpha^+(\Phi_\alpha^+)^{-1}. \quad (41)$$

3.2 Flow equations in a twisted basis

For later use, let us define the “twisted” operators as follows

$$\hat{\chi}_+^\sharp := a\hat{\chi}_+ + b\hat{\chi}_-, \quad \hat{\chi}_-^\sharp := c\hat{\chi}_+ + d\hat{\chi}_-, \quad \hat{\bar{\chi}}_+^\sharp := d\hat{\bar{\chi}}_+ + c\hat{\bar{\chi}}_-, \quad \hat{\bar{\chi}}_-^\sharp := b\hat{\bar{\chi}}_+ + a\hat{\bar{\chi}}_-, \quad (42)$$

with $ad - bc = 1$. One can easily show that these twisted operators also satisfy the same anti-commutation relations $\{\hat{\chi}_\pm^\sharp, \hat{\bar{\chi}}_\pm^\sharp\} = \pm 1$. Thus, one can construct generalized coherent states in terms of the *twisted* operators, which are related to an untwisted state (19) as follows:

$$\langle \eta_+^\sharp, \bar{\eta}_-^\sharp | \bar{\chi}_+, \chi_- \rangle = a \exp \left\{ \int_k \left(-\frac{1}{a} \bar{\chi}_+ \eta_+^\sharp - \frac{1}{a} \bar{\eta}_-^\sharp \chi_- + \frac{b}{a} \bar{\chi}_+ \chi_- - \frac{c}{a} \bar{\eta}_-^\sharp \eta_+^\sharp \right) \right\}. \quad (43)$$

Note that since the twisting does not involve the annihilation and the creation operators, the Fock vacuum $|0, 0\rangle$ (see Appendix B) is invariant under the twist. Let us parametrize the wave-functional (32) in terms of the twisted basis, as follows

$$\Psi[\chi_+^\sharp, \bar{\chi}_-^\sharp] := \langle \chi_+^\sharp, \bar{\chi}_-^\sharp | \hat{U}(\epsilon, \epsilon_0) | UV \rangle := \exp \left\{ - \int_k \left[\bar{\chi}_-^\sharp F^\sharp \chi_+^\sharp + \bar{\chi}_-^\sharp J_-^\sharp + \bar{J}_+^\sharp \chi_+^\sharp + C^\sharp \right] \right\}. \quad (44)$$

Using the inner product (43), the relation

$$\Psi[\chi_+^\sharp, \bar{\chi}_-^\sharp] = \int \mathcal{D}^4 \chi \langle \chi_+^\sharp, \bar{\chi}_-^\sharp | \bar{\eta}_+, \eta_- \rangle e^{\int_k (\bar{\eta}_+ \eta_+ + \bar{\eta}_- \eta_-)} \Psi[\eta_+, \bar{\eta}_-], \quad (45)$$

and performing the integration over χ , we obtain the following relations

$$F^\sharp = (c + dF)(a + bF)^{-1}, \quad \bar{J}_+^\sharp = \bar{J}_+(a + bF)^{-1}, \quad J_-^\sharp = (a + bF)^{-1} J_-. \quad (46)$$

The flow equation for F^\sharp , by using (35), takes the form

$$\begin{aligned} \sqrt{g^{zz}} \partial_\epsilon F^\sharp &= F^\sharp \left(i\gamma^\mu K_\mu (a^2 + b^2) + 2mab \right) F^\sharp - (ac + bd) (F^\sharp i\gamma^\mu K_\mu + i\gamma^\mu K_\mu F^\sharp) \\ &\quad - 2m(ad + bc) F^\sharp + i\gamma^\mu K_\mu (c^2 + d^2) + 2mcd. \end{aligned} \quad (47)$$

A particularly convenient choice is given by $a = b = -c = d = 1/\sqrt{2}$, for which we obtain

$$\sqrt{g^{zz}} \partial_\epsilon F^\sharp = F^\sharp \left(i\gamma^\mu K_\mu + m \right) F^\sharp + i\gamma^\mu K_\mu - m. \quad (48)$$

In this basis, the standard and alternative fixed points of AdS_{d+1} , which correspond to $F = 0$ and $F = \infty$, respectively (see Section 4), are at $F^\sharp = \pm 1$. A special case of Equation (48), for pure AdS geometries and for the homogeneous mode ($k_\mu = 0$), is in agreement with the corresponding flow equation in [18].

Application of the twisting to the rotationally symmetric case is straightforward.

4 Fermionic flows in AdS_{d+1}

Exact solution In AdS_{d+1} with a constant electric field $A_t = \mu$, the flow equation (37) for F becomes

$$\epsilon \partial_\epsilon F = F \epsilon \left(i \gamma^{\hat{\mu}} K_\mu \right) F - 2mF + \epsilon \left(i \gamma^{\hat{\mu}} K_\mu \right), \quad (49)$$

an exact solution of which is given by

$$F = \frac{i \gamma^{\hat{\mu}} K_\mu}{K} \frac{I_{-\nu_+}(\epsilon K) + \chi(k_\mu) I_{\nu_+}(\epsilon K)}{I_{-\nu_-}(\epsilon K) + \chi(k_\mu) I_{\nu_-}(\epsilon K)}, \quad (50)$$

$$K := \sqrt{k^2 - (\omega + q\mu)^2},$$

where $\chi(k_\mu)$ is an integration constant, and $\nu_\pm := m \pm \frac{1}{2}$.

Flows from the alternative to standard fixed point In order to visualize the flow, let us now mimic the discussion in the bosonic case and make a derivative expansion of the double-trace coupling.²¹ From (33), we see that in standard quantization F appears in the effective action as $\bar{O}_- F O_+$. Viewing F as a function of ϵ and $\bar{K}_\mu = \epsilon K_\mu$, the flow equation (49) becomes

$$\epsilon \partial_\epsilon F = F \left(i \gamma^{\hat{\mu}} \bar{K}_\mu \right) F - \left(2m + \bar{K}_\mu \frac{\partial}{\partial \bar{K}_\mu} \right) F + i \gamma^{\hat{\mu}} \bar{K}_\mu. \quad (51)$$

Expanding F as²²

$$F(\epsilon, \bar{K}_\mu) = f_0(\epsilon) + i \gamma^{\hat{\mu}} \bar{K}_\mu f_1(\epsilon) + \bar{K}^2 f_2(\epsilon) + \dots, \quad (52)$$

leads to the flow equations

$$\begin{aligned} \epsilon \partial_\epsilon f_0 &= -2m f_0, \\ \epsilon \partial_\epsilon f_1 &= f_0^2 - (1 + 2m) f_1 + 1, \\ &\dots \end{aligned} \quad (53)$$

from which it is clear that there is a fixed point at $(f_0, f_1) = (0, 1/(2m + 1))$. From the eigenvalues of the flow, the fixed point is attractive in the IR; in Section 7.2 we show that the CFT defined by it indeed corresponds to standard quantization.

In alternative quantization, the double trace term in the effective action is (cf. (34)) is $\bar{O}_+ F^{-1} O_-$. Expanding the coupling F^{-1} as

$$F(\epsilon, \bar{K}_\mu)^{-1} = g_0(\epsilon) + i \gamma^{\hat{\mu}} \bar{K}_\mu g_1(\epsilon) + \bar{K}^2 g_2(\epsilon) + \dots, \quad (54)$$

and following the above discussion, we then obtain the flow equations²³

$$\begin{aligned} \epsilon \partial_\epsilon g_0 &= 2m g_0, \\ \epsilon \partial_\epsilon g_1 &= -g_0^2 - (1 - 2m) g_1 - 1, \\ &\dots \end{aligned} \quad (55)$$

with a fixed point at $(g_0, g_1) = (0, 1/(2m - 1))$. From the eigenvalues of the flow, the fixed point is clearly repulsive; in Section 7.2 we show that this UV fixed point indeed corresponds to alternative quantization.

²¹In essence, this focuses the attention to a particular class of flows which can be obtained from (50) by choosing the integration constant $\chi(k_\mu)$ such that only integral powers of $\gamma^{\hat{\mu}} K_\mu$ are present in the expansion.

²²In (52) we consider Lorentz invariant double trace deformations; deformations and fixed points which are Lorentz non-invariant are considered at the end of this section (see (63)).

²³Generically, $g_0 = f_0^{-1}$, $g_1 = -f_1 f_0^{-2}$, and so on.

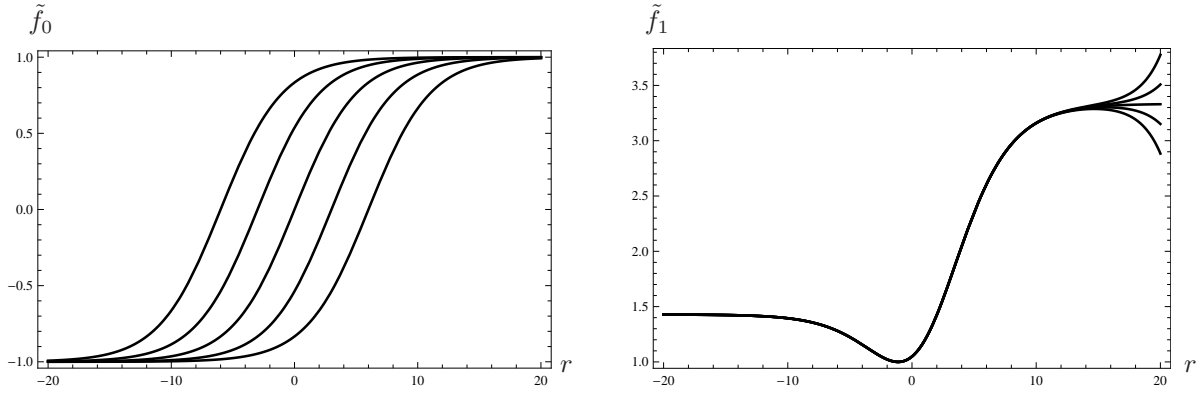


Figure 2: Flows of \tilde{f}_0 and \tilde{f}_1 in AdS_{d+1} as a function of $r = -\log z$ for $m = 0.2$. The UV is located at $r = \infty$ and the IR at $r = -\infty$. In the left panel, different curves correspond to different values of r_0 in (60). In the right panel, r_0 has been chosen to be 1, and different curves correspond to different values of C .

In order to be able to see flows from the alternative to the standard fixed point, it is practical to make the change of variables (corresponding to a specific choice (48) of the twisted basis (42))

$$\tilde{F} = \frac{F - 1}{F + 1}, \quad (56)$$

after which the flow equation becomes

$$\epsilon \partial_\epsilon \tilde{F} = \tilde{F} \left(i\gamma^{\hat{\mu}} \bar{K}_\mu + m \right) \tilde{F} - \bar{K}_\mu \frac{\partial \tilde{F}}{\partial \bar{K}_\mu} + i\gamma^{\hat{\mu}} \bar{K}_\mu - m. \quad (57)$$

Expanding \tilde{F} as

$$\tilde{F}(\epsilon, \bar{K}_\mu) = \tilde{f}_0(\epsilon) + i\gamma^{\hat{\mu}} \bar{K}_\mu \tilde{f}_1(\epsilon) + \bar{K}^2 \tilde{f}_2(\epsilon) + \dots, \quad (58)$$

we obtain the flow equations

$$\begin{aligned} \epsilon \partial_\epsilon \tilde{f}_0 &= m \left(\tilde{f}_0^2 - 1 \right), \\ \epsilon \partial_\epsilon \tilde{f}_1 &= \tilde{f}_0^2 + \left(2m\tilde{f}_0 - 1 \right) \tilde{f}_1 + 1, \\ &\dots, \end{aligned} \quad (59)$$

whose fixed points are at $(\tilde{f}_0, \tilde{f}_1) = (\pm 1, 2/(1 \mp 2m))$ with minus and plus signs corresponding to standard and alternative quantizations, respectively. These equations can be solved exactly ($r := -\log z$):

$$\begin{aligned} \tilde{f}_0 &= \tanh(m(r - r_0)), \\ \tilde{f}_1 &= \frac{C e^{r-r_0}}{4 \cosh^2(m(r - r_0))} + \frac{\tanh^2(m(r - r_0)) + 4m \tanh(m(r - r_0)) + 1}{1 - 4m^2}, \end{aligned} \quad (60)$$

where r_0 and C are integration constants. In Figure 2, we show typical flows of \tilde{f}_0 and \tilde{f}_1 , while Figure 3 shows the flow in the \tilde{f}_0 - \tilde{f}_1 plane. As can be seen, the flows are from the alternative fixed point in the UV to the standard fixed point in the IR.

k^{-1} mode If one expands the exact solution (50) for F directly, the expansion generically starts at the order k^{-1} , in the sense that it has the form

$$F(\epsilon, \bar{K}_\mu) = \left(i\gamma^{\hat{\mu}} \bar{K}_\mu \right)^{-1} f_{-1}(\epsilon) + f_0(\epsilon) + i\gamma^{\hat{\mu}} \bar{K}_\mu f_1(\epsilon) + \bar{K}^2 f_2(\epsilon) + \dots \quad (61)$$

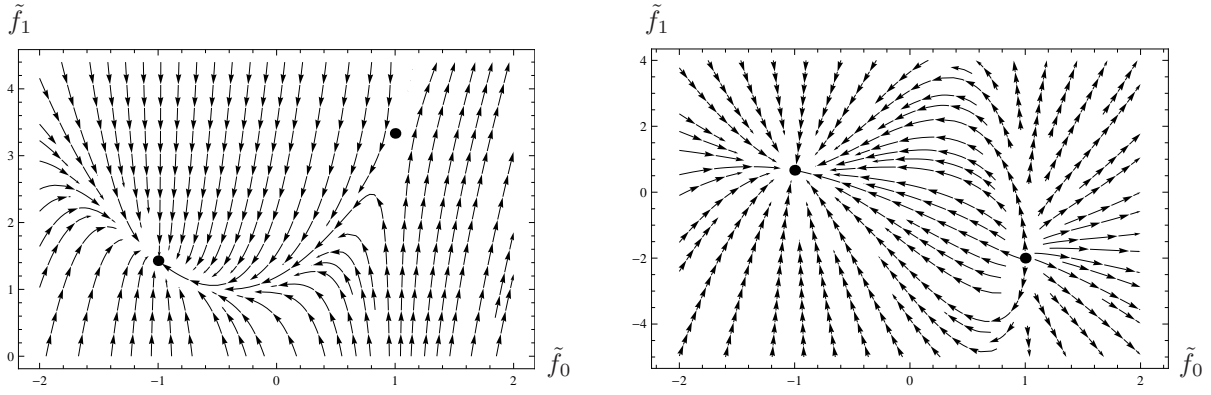


Figure 3: RG flows in the \tilde{f}_0 - \tilde{f}_1 plane for AdS_{d+1} and $m = 0.2$ (left panel) and $m = 1$ (right panel). In the left panel, the two fixed points, at $\tilde{f}_0 = \mp 1$, correspond to standard and alternative quantizations, respectively. In the right panel, the value of m is outside the Klebanov-Witten window and only the standard fixed point defines a quantum theory.

The introduction of a non-local operator like $k^{-1}\overline{O}_-O_+$ near a fixed point is problematic. Fortunately, it is consistent to turn this operator off — it would only start flowing if the theory was perturbed by it initially. This can be seen directly from the resulting flow equations

$$\begin{aligned}\epsilon\partial_\epsilon f_{-1} &= f_{-1}^2 + (1 - 2m)f_{-1}, \\ \epsilon\partial_\epsilon f_0 &= 2(f_{-1} - m)f_0, \\ &\dots\end{aligned}\tag{62}$$

Nevertheless, one might wonder whether these flows, which arise naturally from considering the evolution of the wavefunctional Ψ in the bulk, have an interpretation in (a local) field theory. The double-trace deformation $k^{-1}\overline{O}_-O_+$ can be viewed as arising from integrating out an emergent fermion (see Section 6). The possibility remains to refrain from doing such an integration and study the ‘bigger’ field theory, obtained by coupling the original one (corresponding to standard quantization) to the emergent fermion. This approach will be described in more detail in Section 6 where we will encounter the appearance of emergent fermions at the Fermi surface. After canonically normalizing the kinetic term of the emergent fermion ($\tilde{\chi}_+, \tilde{\chi}_-$) in (82), we obtain that the coupling of the emergent fermion to the sector corresponding to standard quantization is proportional to $\sqrt{f_{-1}}$. At the standard and alternative fixed points, f_{-1} is equal to zero, so that the emergent fermion decouples from the system.

In addition to the standard and alternative fixed points, at $(f_{-1}, f_0) = (0, 0)$ and $(f_{-1}, f_0) = (0, \infty)$, the flow equations (62) have two other fixed points at $(f_{-1}, f_0) = (2m - 1, 0)$ and $(f_{-1}, f_0) = (2m - 1, \infty)$. These two new fixed points are only there because we have coupled the original theory to an emergent fermion. We can visualize flows between these fixed points by making the change of variable $a_0 = (f_0 - 1)/(f_0 + 1)$. As can be seen from Figure 4, outside the Klebanov-Witten window, *i.e.* for $m > 1/2$, the two new fixed points are unstable, so that all flows end up at the standard fixed point. However, for $0 \leq m \leq 1/2$, one of the new fixed points becomes IR stable. At this IR fixed point, the coupling between the emergent fermion and the standard quantization sector is non-zero.

Non-relativistic deformations For simplicity, let us focus on $d = 3$. Consider (49) and make the ansatz $F = i\gamma^{\hat{\mu}}F_{\hat{\mu}} + f$. Concentrating on the homogeneous modes (*i.e.* putting $k = \omega = 0$), we find

$$\epsilon\partial_\epsilon F_{\hat{\mu}} = -2mF_{\hat{\mu}}, \quad \epsilon\partial_\epsilon f = -2mf.\tag{63}$$

Non-zero $F_{\hat{\mu}}$ violates Lorentz invariance. We see that there are new fixed points given by $F_{\hat{\mu}} = 0$ or ∞ for various μ . Starting out with one of these components $F_{\hat{\mu}}$ equal to infinity in the UV, *i.e.* at a Lorentz-violating fixed point, one however always ends up flowing to the usual IR fixed point given by standard quantization, where Lorentz symmetry is restored. Similar flows from Lorentz non-invariant fixed points to Lorentz invariant fixed points occur in discussions involving Lifshitz theories, *e.g.*, in [44] in the context of gravity, and in [45, 46] in the context of field theory.

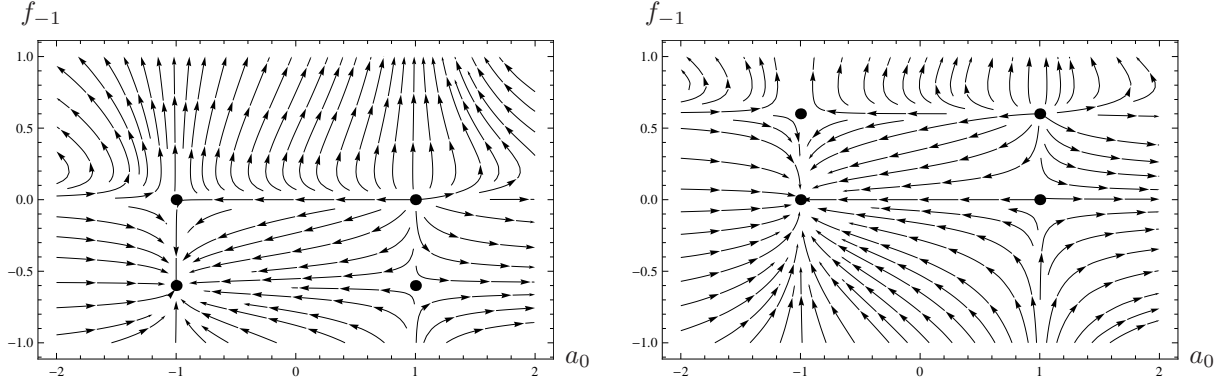


Figure 4: RG flows in the a_0 - f_{-1} plane for AdS_{d+1} and $m = 0.2$ (left panel) and $m = 0.8$ (right panel).

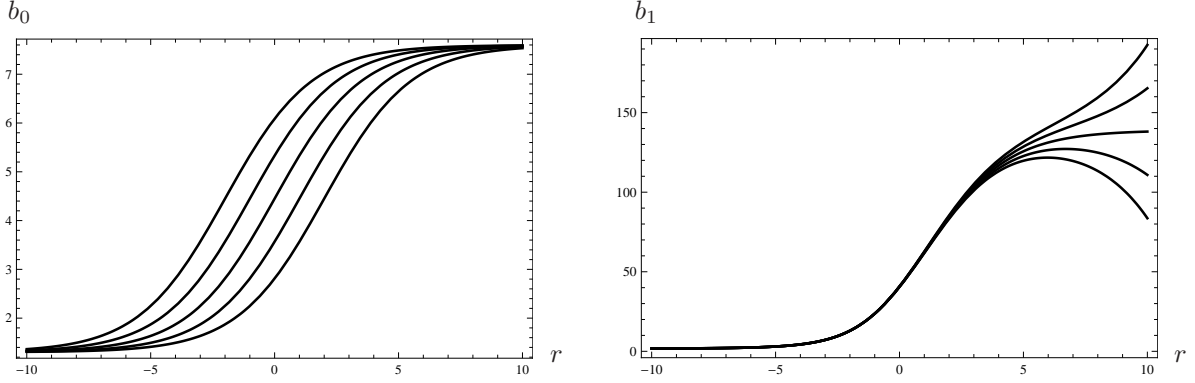


Figure 5: Flows of b_0 and b_1 in the near horizon limit as a function of $r = -\log \zeta$ for $m = 1$, $k = 1$, $q e_d = 0.5$, and $z_* = 1$, making $\nu_k = 0.3$. The UV of the AdS_2 region is at $r = \infty$, while its IR is at $r = -\infty$.

5 Fermionic flows in the extremal charged black hole background

In this section, we will study flows in the extremal charged black hole background given by (9). Since this background is rotationally symmetric, it is practical to use the form of the flow equation (39) given in Section 3.1.1, which was obtained by making a convenient choice of Dirac matrices. Let us concentrate on the case $\alpha = 1$,²⁴ in the following omitting the α -index of F_α . We have that

$$\sqrt{H} \epsilon \partial_\epsilon F = \left(\frac{\epsilon(\omega + q A_t)}{\sqrt{H}} - \epsilon k \right) F^2 - 2mF + \frac{\epsilon(\omega + q A_t)}{\sqrt{H}} + \epsilon k. \quad (64)$$

In the far UV, the ECBH background is given by AdS_{d+1} with a constant electric field, which is the case we studied in the last section. Let us start by analyzing the flow equation for F in the near horizon limit.

5.1 Near horizon limit

The background (9) has an emergent $\text{AdS}_2 \times \mathbb{R}^{d-1}$ geometry in the near horizon limit. To see this [21], define new coordinates ζ and τ through ($R_2 = 1/\sqrt{d(d-1)}$)

$$z - z_* = -\frac{\lambda R_2^2 z_*^2}{\zeta}, \quad t = \lambda^{-1} \tau, \quad \lambda \ll 1, \quad (65)$$

²⁴The flow equation for F_2 is the same after making the switch $k \rightarrow -k$.

so that near the horizon

$$\begin{aligned} ds^2 &= \frac{R_2^2}{\zeta^2} (d\zeta^2 - d\tau^2) + \frac{1}{z_*^2} dx_2^2, \\ A_\tau &= \lambda^{-1} A_t = \frac{R_2^2 \mu z_*}{\zeta} =: \frac{e_d}{\zeta}. \end{aligned} \quad (66)$$

Furthermore, in this limit we will hold $\Omega := \lambda^{-1} \omega$ fixed, thereby focusing on low frequencies ω .

In this limit, the flow equation for F becomes

$$\zeta \partial_\zeta F = (\zeta \Omega + qe_d - R_2 z_* k) F^2 - 2R_2 m F + \zeta \Omega + qe_d + R_2 z_* k. \quad (67)$$

Consider the case $\Omega = 0$. Then the solution is given by ($r = -\log \zeta$)

$$\begin{aligned} F &= \frac{R_2 m + \nu_k \tanh(\nu_k (r - r_0))}{qe_d - R_2 z_* k}, \\ \nu_k &:= \sqrt{R_2^2 (m^2 + z_*^2 k^2) - q^2 e_d^2}, \end{aligned} \quad (68)$$

where, for real ν_k , r_0 is an integration constant that sets the scale of where the kink is (see Figure 5). This solution is a good approximation as long as $\zeta \Omega \ll 1$. This means that even for small Ω , deviations will eventually be seen in the deep IR. For imaginary ν_k , the solution becomes oscillatory, signalling that we are inside what is referred to as the oscillatory region [20].

Viewing F as a function of ζ , k , and $\bar{\Omega} = \zeta \Omega$, we have that

$$\zeta \partial_\zeta F = (\bar{\Omega} + qe_d - R_2 z_* k) F^2 - \left(2R_2 m + \bar{\Omega} \frac{\partial}{\partial \bar{\Omega}} \right) F + \bar{\Omega} + qe_d + R_2 z_* k. \quad (69)$$

Expanding

$$F(\zeta, \bar{\Omega}, k) = \sum_{n=0}^{\infty} b_n(\zeta, k) \bar{\Omega}^n, \quad (70)$$

we obtain

$$\begin{aligned} \zeta \partial_\zeta b_0 &= (qe_d - R_2 z_* k) b_0^2 - 2R_2 m b_0 + qe_d + R_2 z_* k, \\ \zeta \partial_\zeta b_1 &= 2(qe_d - R_2 z_* k) b_0 b_1 + b_0^2 - (1 + 2R_2 m) b_1 + 1, \\ &\dots \end{aligned} \quad (71)$$

Example of flows for b_0 and b_1 are shown in Figure 5.

Merger and disappearance of fixed points Figure 6 shows flow diagrams in the $b_0 - b_1$ plane. There are two fixed points, given by $b_0 = \frac{R_2 m \pm \nu_k}{qe_d - R_2 z_* k}$. As one approaches the oscillatory region, where ν_k becomes imaginary, the two fixed points approach each other until they merge and disappear (become imaginary). Such a merger of fixed points is reminiscent of similar discussions in [25].

5.2 Flows in the full ECBH background

Let us now study flows in the full ECBH background, i.e connecting the two asymptotic regions studied in the previous subsections. After changing variables to (see (56))

$$\tilde{F} = \frac{F - 1}{F + 1}, \quad (72)$$

the flow equation (64) becomes

$$\sqrt{H} \epsilon \partial_\epsilon \tilde{F} = \left(m + \frac{\epsilon(\omega + qA_t)}{\sqrt{H}} \right) \tilde{F}^2 - 2\epsilon k \tilde{F} - m + \frac{\epsilon(\omega + qA_t)}{\sqrt{H}}. \quad (73)$$

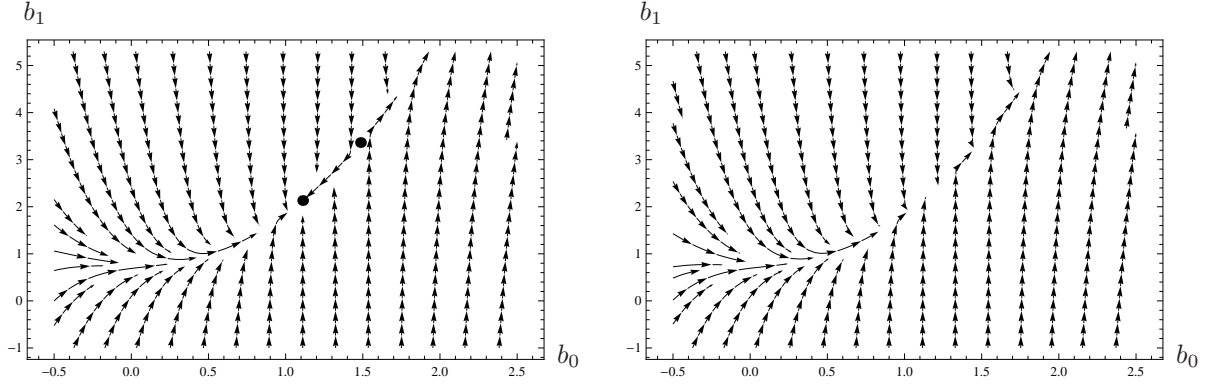


Figure 6: The left panel shows the RG flow in the $b_0 - b_1$ -plane for the near horizon limit, and $d = 3$, $m = 0.4$, $k = 0.1$, $qe_d = 1/6$, $z_* = 1$, making $\nu_k = 0.02$. The right panel shows the same, but for $m = 0.395$ (all other parameters unchanged), making $\nu_k = 0.01i$ imaginary.

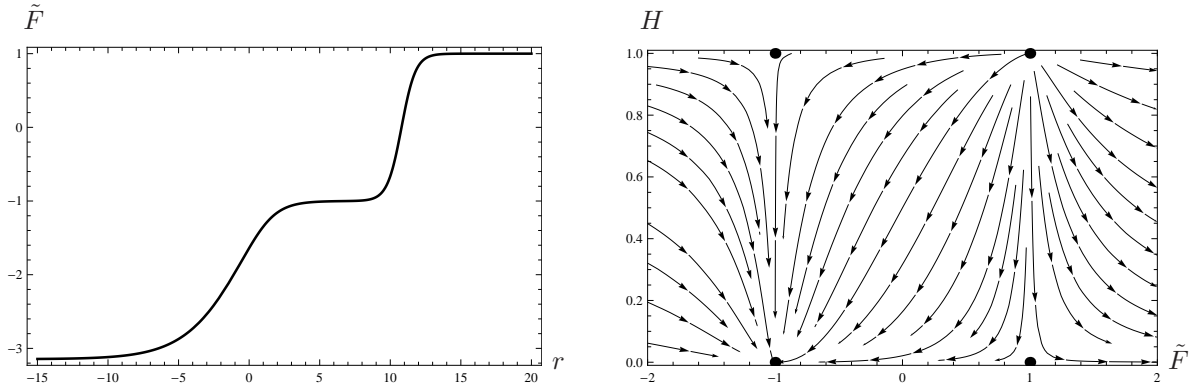


Figure 7: RG flows in the ECBH background and $d = 3$, $m = 1$, $k = 0$, $\omega = 0$. The left panel shows \tilde{F} as a function of $r = \log(z^{-1} - z_*^{-1})$ (so that the UV is at $r = \infty$ while the horizon is at $r = -\infty$) for $q\mu = -2$, $z_* = 1$. The right panel shows the RG flow in the \tilde{F} - H plane for $q = 0$; as in the bosonic case (Figure 1), there are two fixed points in the AdS_4 region ($H \sim 1$) and two fixed points in the AdS_2 region ($H \sim 0$).

The left panel of Figure 7 shows a solution to this equation, which starts by flowing from the alternative to the standard fixed point of the AdS_4 region, and then finally reaches the standard fixed point of the AdS_2 near horizon region. The the right panel of the same figure shows flows in the \tilde{F} - H plane obtained for $q = 0$, and $k = \omega = 0$ (thereby focusing on the homogeneous mode), the fixed points of which are at $(\tilde{F}, H) = (\pm 1, 1)$ and $(\tilde{F}, H) = (\pm 1, 0)$. The flow of H is interpreted, as in Section 2 (see Section 7 for further details).

6 Emergent fermions and semi-holography

The semi-holographic approach and goal of this section The main goal of this section is to understand the semi-holographic approach to fermionic field theories [19] from the Wilsonian perspective followed in this paper. In this approach, one considers a weakly coupled fermionic field χ_a coupled to fermionic fields Ψ_a ²⁵ belonging to a strongly interacting sector²⁶

$$S = S_{\text{strong}} + \int dt \int d^d k \left[\chi^\dagger(k) (i\partial_t - \varepsilon(k) + \mu q) \chi(k) + g_k \chi^\dagger(k) \Psi(k) + g_k^* \Psi^\dagger(k) \chi(k) \right]. \quad (74)$$

²⁵The label a is a Dirac spinor index; the notation Ψ for the field theory operator should not be confused with the wavefunction Ψ , e.g., (77).

²⁶ $\varepsilon(k)$ in (74) represents the single particle energy of the field χ_a .

While χ_a is described by standard field theory methods, the strongly coupled field Ψ_a is described holographically using AdS/CFT. In particular, by usual field theory arguments one can derive [19]

$$G_g(\omega, k) = \frac{1}{G_0(\omega, k)^{-1} - |g_k|^2 \mathcal{G}_0(\omega, k)}, \quad (75)$$

where G and \mathcal{G} are the Feynman propagators for the χ and the Ψ fields, respectively. The subscript $_0$ indicates the decoupled limit where $g_k = 0$; G_g denotes the χ propagator in presence of the coupling. The two-point function $\mathcal{G}_0(k)$ from the strongly coupled sector is derived holographically using an AdS dual. The influence of the strong sector can clearly have interesting consequences for Equation (75). An example is the non-Fermi-liquid behaviour described in [21].

In [19] a qualitative origin of the fields χ and Ψ is described in the context of the charged AdS black hole (9) discussed in this paper. In particular, Ψ are operators in the field theory dual to the near-horizon $\text{AdS}_2 \times \mathbb{R}^2$ region, whereas χ is supposed to be a ‘domain wall fermion’ near the AdS_2 boundary. In what follows, we will describe the precise origin of the field χ , from the Wilsonian RG perspective, as an emergent fermion. A hint of such an approach appeared earlier in [2].

6.1 Appearance of massless fermions and derivation of semi-holography

We start by applying (30) to the charged black hole background (9), which has an emergent near-horizon $\text{AdS}_2 \times \mathbb{R}^{d-1}$ region. We put an intermediate cut-off ϵ at the AdS_2 boundary. In other words, we split the bulk transition amplitude from the black hole horizon to the AdS_{d+1} boundary into two parts: (a) from the horizon to the AdS_2 boundary, and (b) from the AdS_2 boundary to the AdS_{d+1} boundary. We rewrite (30) as

$$\langle \text{IR} | \widehat{U}(\epsilon_{\text{IR}}, \epsilon_0) | \text{UV} \rangle = \int D^4 \tilde{\chi} \langle \text{IR} | \widehat{U}(\epsilon_{\text{IR}}, \epsilon) | \tilde{\chi}_+, \tilde{\chi}_- \rangle e^{\int_k \tilde{\chi}_+ \tilde{\chi}_+ + \tilde{\chi}_- \tilde{\chi}_-} \Psi[\tilde{\chi}_+, \tilde{\chi}_-], \quad (76)$$

$$\Psi[\tilde{\chi}_+, \tilde{\chi}_-] = \langle \tilde{\chi}_+, \tilde{\chi}_- | \widehat{U}(\epsilon, \epsilon_0) | \text{UV} \rangle, \quad (77)$$

where we have added a ‘tilde’ to χ to indicate the possibility of a different basis of coherent states at the intermediate cut-off (we will see below in (103) that such a change of basis is induced when we enter the AdS_2 region). Note that $\Psi[\tilde{\chi}_+, \tilde{\chi}_-]$ does not have to be a transition amplitude; it is enough to assume that a Gaussian wave-functional is given at the cut-off surface ϵ . Later, we adopt this viewpoint, regarding the AdS_2 boundary as the cut-off surface.

By using the quadratic ansatz (32) for the wavefunction Ψ ,

$$\Psi[\tilde{\chi}_+, \tilde{\chi}_-] := \exp \left\{ -\frac{1}{\kappa^2} \int_k \left[\tilde{\chi}_-(k) \tilde{F}(k, \epsilon) \tilde{\chi}_+(k) + \tilde{\chi}_-(k) \tilde{J}_-(k, \epsilon) + \tilde{J}_+(k, \epsilon) \tilde{\chi}_+(k) + \tilde{C}(\epsilon) \right] \right\}, \quad (78)$$

and performing the $\int D^4 \tilde{\chi}$ integral, we arrive, as in (33), at the following field theory at the intermediate cut-off

$$\left\langle \exp \int_k \left[-\overline{O}_-(k) \tilde{F}(k, \epsilon) O_+(k) + \tilde{J}_+(k, \epsilon) O_+(k) + \overline{O}_-(k) \tilde{J}_-(k, \epsilon) \right] \right\rangle_\epsilon^{\text{std}}. \quad (79)$$

Now, suppose after solving the flow equations for \tilde{F} up to $z = \epsilon$, we find that $\tilde{F}(k, \epsilon)$ develops a single pole at some $k_\mu = k_{*\mu}$ ²⁷

$$\tilde{F}(k, \epsilon) \sim \frac{1}{c_\mu(k - k_*)_\mu}. \quad (80)$$

²⁷ Let us see the structure of c_μ . Here let us adopt the representation (15) or (16) of the Dirac matrices, which is different from the representation in the next subsection. From the equation of motion, we find $K^\nu K_\nu \Phi_\mp = -i\gamma^\mu K_\mu (\pm \partial_z - m\sqrt{g_{zz}}) \Phi_\pm$. Since we can show that $\Phi_- (\Phi_+)^{-1}$ satisfies the flow equation of F , F is proportional to $\gamma^\mu K_\mu$ and thus $c_\mu(k - k_*)_\mu$ can be expressed as $A_\mu \gamma^\mu$. Now, since A_μ is a linear function of $k - k_*$, A_μ can be rewritten as $A_\mu = A_{\mu\nu}(k - k_*)_\nu$ with some tensor $A_{\mu\nu}$. Using the rotational symmetry in the spatial directions, we find $A_{ti} = 0$, $A_{ij} = B\delta_{ij}$. We thus find $c_t = B_{tt}\gamma^t$, $c_i = B\gamma^i$.

We will see below that this indeed happens in the case of the extremal charged black hole background, where $k_{*\mu}$ in (80) is given by $k_*^0 = \omega = 0$ and $\vec{k}_* = k_F \vec{k}/|\vec{k}|$, with k_F the Fermi momentum, and ϵ represents the AdS_2 boundary. Substituting the above form of \tilde{F} into the IR generating functional (79), we find the following non-local expression,

$$\left\langle\left\langle \exp \left(\int_k -\overline{O}_- \frac{a_1}{c_\mu(k-k_*)_\mu} O_+ + \dots \right) \right\rangle\right\rangle_\epsilon^{\text{std}}. \quad (81)$$

where \dots denotes terms regular in $(k-k_*)_\mu$. The appearance of such non-local operators is a typical signal of appearance of massless modes in the system which we have somehow integrated out. To figure out what might have happened, recall that (79) was arrived at by performing the integral $\int D^4 \tilde{\chi}$ in (76). Suppose, instead, we leave the integration over $\tilde{\chi}_+, \tilde{\chi}_-$ *undone*; in that case the field theory correlator (79) is replaced by²⁸

$$\int D\tilde{\chi}_+ D\tilde{\chi}_- \left\langle\left\langle \exp \int_k [\tilde{\chi}_+ O_+ + \overline{O}_- \tilde{\chi}_-] \right\rangle\right\rangle_\epsilon^{\text{std}} \exp \int_k [\tilde{\chi}_+ \tilde{F}^{-1} \tilde{\chi}_- - \tilde{J}_+ \tilde{F}^{-1} \tilde{\chi}_- - \tilde{\chi}_+ \tilde{F}^{-1} \tilde{J}_-]. \quad (82)$$

Note that $\tilde{\chi}_+, \tilde{\chi}_-$ in (76), which do not couple to O_+, \overline{O}_- , have been integrated out. Writing $\langle\langle (\dots) \rangle\rangle_\epsilon := \int DM|_{\Lambda(\epsilon)} (\dots) e^{S[M, \epsilon]}$, where M is understood to be some large N matrix field with action $S[M]$, (82) can be rewritten as²⁹

$$\int DM D\tilde{\chi}_+ D\tilde{\chi}_- e^{S_{\text{total}}[\epsilon]}, \quad S_{\text{total}}[\epsilon] := S[M, \epsilon] - \int_k a_1^{-1} \tilde{\chi}_+(k) [c_\mu(k-k_*)_\mu + \dots] \tilde{\chi}_-(k) + \int_k (\tilde{\chi}_+ \mathbb{O}_+ + \overline{\mathbb{O}}_- \tilde{\chi}_-), \quad (83)$$

where we defined modified the new composite fields

$$\mathbb{O}_+ := O_+ - \tilde{F}(\epsilon)^{-1} \tilde{J}_-(\epsilon), \quad \overline{\mathbb{O}}_- := \overline{O}_- - \tilde{J}_+(\epsilon) \tilde{F}(\epsilon)^{-1}. \quad (84)$$

Thus, S_{total} is the new action at the AdS_2 boundary which includes the emergent massless fermions $\tilde{\chi}_+, \tilde{\chi}_-$; the action has the form of the semi-holographic action (74), provided we make the identifications shown in Table 2.

Faulkner-Polchinski	χ_a	$\omega = 0, \vec{k} = k_F$	Ψ_a	S_{strong}
Here	$\tilde{\chi}_+, \tilde{\chi}_-$	$k_{*,\mu}$ (see (80))	$\mathbb{O}_+, \overline{\mathbb{O}}_-$	$S[M, \epsilon]$

Table 2: Emergent fermions from Wilsonian RG. With the identification in this Table, with an appropriate rescaling of $\tilde{\chi}$, our equation (83) reproduces the equation (74) of [19], including a determination of the couplings g_k .

The identification clearly shows that the ‘domain wall’ fermion χ_a of [19], appearing in Equation (74), can be identified with the emergent fermions $\tilde{\chi}_+, \tilde{\chi}_-$ at the AdS_2 boundary.

In the rest of this section, we will explain how the pole structure (80) arises at the AdS_2 boundary, with $k_\mu = k_{*\mu}$ identified with $\omega = 0, |\vec{k}| = k_F$.

6.2 Double trace coupling of the IR theory

We begin by rewriting the ECBH background

$$ds^2 = \frac{R^2}{z^2} (-h(z) dt^2 + dx^i dx^i) + \frac{R^2}{z^2} \frac{dz^2}{h(z)}, \quad A(z) = \mu \left(1 - \frac{z^{d-2}}{z_*^{d-2}} \right) dt, \quad (85)$$

where R is the AdS_{d+1} radius.³⁰ Note that $z = 0$ is the AdS_{d+1} boundary, z_* is the horizon radius, and μ is the chemical potential in the field theory (for details, see [21]). As in the previous section, we consider only the $\alpha = 1$

²⁸Here we dropped the quantum correction $\det \tilde{F}$ and terms constant in $\tilde{\chi}_+, \tilde{\chi}_-$.

²⁹The form of the kinetic term of the emergent fermions in (83) is appropriate near the pole $k = k_*$ with $k_*^0 = \omega = 0, |\vec{k}| = k_F$, the Fermi momentum; if one is interested in physics away from the pole these new fermions can be integrated out, leading to local operators in terms of the original theory.

³⁰In order to clarify the meaning of near horizon limit given later, we exhibit the AdS_{d+1} radius explicitly.

sector (this can be done without loss of generality since the $\alpha = 1$ sector is decoupled from $\alpha = 2$ and these two are related by $k_i \leftrightarrow -k_i$).³¹

Two solution bases of Dirac equations We now relate the double trace coupling in the UV region (the AdS_{d+1} boundary) to the double trace coupling in the IR region (the AdS_2 boundary). For this purpose, we introduce two solution bases [21] of the bulk Dirac equations of motion (138) on the full ECBH background (85),

$$1. \quad (\phi_{\pm}, \varphi_{\pm}) : \phi_{-} \sim z^{-m}, \quad \phi_{+} \sim z^{-m+1}, \quad \varphi_{-} \sim z^{m+1}, \quad \varphi_{+} \sim z^m \quad (z \sim 0), \quad (86)$$

$$2. \quad (\rho_{\pm}, \psi_{\pm}) : \rho_{\pm} \sim \xi^{-\nu_k}, \quad \psi_{\pm} \sim \xi^{\nu_k} \quad (\omega\xi \sim 0), \quad (87)$$

where ν_k is given in (100) or (146), and the new coordinate ξ was introduced as follows:

$$\xi := \frac{1}{d(d-1)} \frac{z_*^2}{z_* - z}, \quad R_2 := \frac{R}{\sqrt{d(d-1)}}. \quad (88)$$

In terms of those solution bases, any solution is expressed as

$$\Phi = A \begin{pmatrix} \phi_{+} \\ \phi_{-} \end{pmatrix} + B \begin{pmatrix} \varphi_{+} \\ \varphi_{-} \end{pmatrix} = C \begin{pmatrix} \rho_{+} \\ \rho_{-} \end{pmatrix} + D \begin{pmatrix} \psi_{+} \\ \psi_{-} \end{pmatrix}. \quad (89)$$

First, let us explain the first solution basis (86). The asymptotic behavior (86) can be found from the explicit solutions (141) to the Dirac equations on the near-boundary AdS_{d+1} geometry; explicitly, ϕ_{\pm} asymptote to $\sqrt{z} I_{-\nu_{\mp}}(zK)$ and φ_{\pm} asymptote to $\sqrt{z} I_{\nu_{\mp}}(zK)$.³² The basis (86) is used for identifying source fields at the AdS_{d+1} boundary.

To understand the second solution basis (87) and to see the role played by ξ , let us define the inner and outer regions (for all details, see [47])

$$\text{the inner region} : 0 < \frac{z_* - z}{z_*} < \lambda, \quad \omega z_* < \lambda, \quad \lambda \ll \mu z_*, \quad (90)$$

$$\text{the outer region} : \lambda' < \frac{z_* - z}{z_*} < \infty, \quad \omega z_* \ll \lambda'. \quad (91)$$

The matching region is defined as the following double scaling limit

$$\text{the matching region} : \omega z_* \ll \lambda' < \frac{z_* - z}{z_*} < \lambda \ll \mu z_*. \quad (92)$$

Then, the spacetime metric and the gauge field in the inner region in the $\lambda \rightarrow 0$ limit become (cf. (66))³³

$$ds^2 = \frac{R_2^2}{\zeta^2} (-d\tau^2 + d\zeta^2) + \frac{R^2}{z_*^2} dx^i dx^i, \quad A = \frac{e_d}{\zeta} d\tau, \quad \zeta := \lambda\xi, \quad \tau := \lambda t, \quad (93)$$

where ζ, τ are kept finite in $\lambda \rightarrow 0$, and e_d is proportional to the gauge coupling constant (for the explicit definition, see [21]). This shows that ξ (or ζ) is the radial coordinate of the $\text{AdS}_2 \times \mathbb{R}^{d-1}$.

The asymptotic behavior of the second solution basis (87) can be found from the near- AdS_2 -boundary behavior of solutions to the Dirac equations of motion in the inner region in the leading order in λ , given by (146) with (138), which are the Dirac equations in the $\text{AdS}_2 \times \mathbb{R}^{d-1}$ background (93). In fact, this inner region Dirac equation yields an analytic expression of the solution basis (ρ_{\pm}, ψ_{\pm}) in terms of the Whittaker functions (see also [48])³⁴

$$\begin{aligned} \begin{pmatrix} \rho_{+} \\ \rho_{-} \end{pmatrix} &= (2i\omega)^{\nu_k - \frac{1}{2}} (1 - ic_k)^{-1} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} c_k M_{-iqe_d - \frac{1}{2}, -\nu_k}(2i\omega\xi) \\ M_{-iqe_d + \frac{1}{2}, -\nu_k}(2i\omega\xi) \end{pmatrix}, \\ \begin{pmatrix} \psi_{+} \\ \psi_{-} \end{pmatrix} &= (2i\omega)^{-\nu_k - \frac{1}{2}} (1 - ic_k)^{-1} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} c_k M_{-iqe_d - \frac{1}{2}, \nu_k}(2i\omega\xi) \\ M_{-iqe_d + \frac{1}{2}, \nu_k}(2i\omega\xi) \end{pmatrix}, \quad c_k := \frac{-\nu_k + iqe_d}{R_2(m + kz_*R^{-1})}. \end{aligned} \quad (96)$$

³¹ For the definition and properties of α , see Appendix C.

³² (ϕ_{-}, φ_{-}) are normalized so that the first coefficients of the Taylor expansions of them are 1, while (ϕ_{+}, φ_{+}) are defined by (138) from (ϕ_{-}, φ_{-}) and thus are not normalized so that any solution be expressed as (89).

³³ The two coordinates ζ and ξ are related by $\Omega\zeta = \xi\omega$, where $\Omega = \lambda^{-1}\omega$ is kept fixed in the limit $\lambda \rightarrow 0$ (see also Appendix C.1.2).

³⁴ The Whittaker functions $M_{\kappa, \mu}(z)$, defined by

$$M_{\kappa, \mu}(z) := z^{\mu + \frac{1}{2}} e^{-z/2} \sum_{n=0}^{\infty} \frac{\Gamma(2\mu + 1) \Gamma(\mu - \kappa + n + \frac{1}{2})}{\Gamma(2\mu + n + 1) \Gamma(\mu - \kappa + \frac{1}{2})}, \quad (94)$$

In addition, this inner region solution basis can be extended to the outer region by matching the above inner region solutions with the solutions to the outer region Dirac equation (for details, see [21, 47]). This is the second solution (87) *defined all over the ECBH background*.

Here let us make an important remark on the matching region. We can show that $\xi\omega \ll 1$ holds for any ξ in the matching region [47]. This means that any point in the matching region is very close to the AdS_2 boundary. Thus, any $\xi=\text{constant}$ surface in the matching region can be interpreted as a surface which regularizes the AdS_2 boundary. We will use this interpretation later when we consider the holography of the inner region with $\text{AdS}_2 \times \mathbb{R}^{d-1}$ metric (93).

Renormalized double trace coupling in the continuum limit at UV Let us consider the renormalized double trace operators in the continuum limit of the UV theory. The double trace coupling flow can be expressed in terms of the classical solution basis (86) using (41),

$$F(z) = \frac{\phi_-(z) + \chi_{\text{UV}}\varphi_-(z)}{\phi_+(z) + \chi_{\text{UV}}\varphi_+(z)}, \quad (97)$$

where χ_{UV} is a constant under the flow and is fixed by the boundary conditions at the AdS_{d+1} boundary. Applying the general formula (171), we can find the asymptotic behavior of the double trace coupling near the boundary $z \sim 0$,

$$\frac{1}{F(z)^{-1} - A_0^{\text{UV}}} \sim \chi_{\text{UV}}^{-1} z^{-2m} \quad (z \sim 0), \quad (98)$$

where A_0^{UV} comes from the counterterm action. If it is given by $A_0^{\text{UV}} = \phi_+/\phi_-$ (see (169)), the renormalized double trace coupling becomes χ_{UV}^{-1} .

Renormalized double trace coupling at IR Let us apply the holography in the matching region. As mentioned above, since any point in the matching region is very close to the AdS_2 boundary, we can expect that any cutoff theory in the matching region can be well approximated by the continuum theory, even though the boundary of $\text{AdS}_2 \times \mathbb{R}^{d-1}$ itself is, strictly speaking, not in the inner (matching) region.

Near the AdS_2 boundary $\omega\xi \sim 0$, the bulk Dirac equation in the background (93) can be approximated by

$$\zeta\partial_\zeta\Phi = U\Phi, \quad U := \begin{pmatrix} mR_2 & \tilde{m}R_2 - qe_d \\ \tilde{m}R_2 + qe_d & -mR_2 \end{pmatrix}, \quad \tilde{m} := \frac{kz_*}{R}. \quad (99)$$

and, as seen from (87), two components of each solution basis have the same powers. Thus, it is ambiguous to choose which component is set to be a non-normalizable mode in formulating the GKP-Witten relations.³⁵ In order to avoid this ambiguity, let us diagonalize the Dirac equation,

$$\zeta\partial_\zeta\tilde{\Phi} = \begin{pmatrix} \nu_k & 0 \\ 0 & -\nu_k \end{pmatrix} \tilde{\Phi}, \quad \tilde{\Phi} := V^{-1}\Phi, \quad V := (\mathbf{v}_+, \mathbf{v}_-), \quad \nu_k := \sqrt{(m^2 + \tilde{m}^2)R_2^2 - q^2e_d^2}, \quad (100)$$

where \mathbf{v}_\pm are eigenvectors of U of eigenvalues $\pm\nu_k$, and V is a 2×2 matrix. Note that we can always have $\det V = 1$ by normalizing \mathbf{v}_\pm appropriately. Thus, the field redefinition $\Phi \rightarrow \tilde{\Phi}$ is nothing but the twisting (42) and hence we rewrite $\tilde{\Phi}$ as Φ^\sharp . In this twist, the four coefficients in (42) are given by³⁶

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = V^{-1}. \quad (101)$$

are solutions to the Whittaker differential equation

$$\left(\partial_z^2 - \frac{1}{4} + \frac{\kappa}{z} - \frac{\mu^2 - \frac{1}{4}}{z^2} \right) M_{\kappa, \mu}(z) = 0, \quad (95)$$

and a pair $(M_{\kappa, \mu}(z), M_{\kappa, -\mu}(z))$ forms a solution basis. Note that (ρ_-, ψ_-) are normalized so that the first coefficients of the Taylor expansions of them are 1, while (ρ_+, ψ_+) are defined by (138) from (ρ_-, ψ_-) so that any solution be expressed as (89).

³⁵In other words, there is an ambiguity in the statement that $|\overline{\chi}_+, \chi_- \rangle$ can be an initial state for the standard quantization (see (19)).

³⁶A convenient choice is given by $a = \frac{1}{\sqrt{2}} \frac{qe_d + \tilde{m}R_2}{\nu_k}$, $b = \frac{1}{\sqrt{2}} \left(1 - \frac{mR_2}{\nu_k} \right)$, $c = -\frac{1}{\sqrt{2}}$, $d = \frac{1}{\sqrt{2}} \frac{mR_2 + \nu_k}{qe_d + \tilde{m}R_2}$.

In terms of the *twisted fields*, it is possible to formulate the GKP-Witten relation of the $\text{AdS}_2 \times \mathbb{R}^{d-1}$ region unambiguously in a parallel manner to Section 3 (in particular (19), (20)).

Now, let us obtain the double trace coupling at the AdS_2 boundary. The expression $F(z)$ of (97), which is defined *all over the ECBH background*, can also be expressed in terms of the second solution basis (87),

$$F(z) = \frac{\phi_-(z) + \chi_{\text{UV}}\varphi_-(z)}{\phi_+(z) + \chi_{\text{UV}}\varphi_+(z)} = \frac{\rho_-(\xi(z)) + \chi_{\text{IR}}\psi_-(\xi(z))}{\rho_+(\xi(z)) + \chi_{\text{IR}}\psi_+(\xi(z))}, \quad z \in (\text{the inner region}) \cup (\text{the outer region}), \quad (102)$$

where $\xi(z)$ is given in (88). Note that F is defined in terms of the original *untwisted* fields as (149). However, the appropriate double-trace coupling flow in the matching region is the twisted one F^\sharp . Plugging (102) into (46), we find

$$F^\sharp(\xi) = \frac{\rho_-^\sharp(\xi) + \chi_{\text{IR}}\psi_-^\sharp(\xi)}{\rho_+^\sharp(\xi) + \chi_{\text{IR}}\psi_+^\sharp(\xi)}, \quad \xi \in (\text{the matching region}), \quad (103)$$

where we defined the twisted solution basis following the twist of the Dirac fields (100),

$$\begin{pmatrix} \rho_+^\sharp(\xi) \\ \rho_-^\sharp(\xi) \end{pmatrix} := V^{-1} \begin{pmatrix} \rho_+(\xi) \\ \rho_-(\xi) \end{pmatrix}, \quad \begin{pmatrix} \psi_+^\sharp(\xi) \\ \psi_-^\sharp(\xi) \end{pmatrix} := V^{-1} \begin{pmatrix} \psi_+(\xi) \\ \psi_-(\xi) \end{pmatrix}. \quad (104)$$

Near the AdS_2 boundary, i.e. using the approximation to Dirac's equation given in (99), the twisted solutions behave as $\rho_-^\sharp \sim \xi^{-\nu_k}$, $\psi_+^\sharp \sim \xi^{\nu_k}$ and $\rho_+^\sharp = \psi_-^\sharp = 0$. Therefore, the twisted double-trace coupling F^\sharp in the matching region behaves as

$$F^\sharp(\xi) \sim \chi_{\text{IR}}^{-1} \xi^{-2\nu_k} \quad (\omega\xi \sim 0), \quad (105)$$

so that the renormalized double trace coupling at the AdS_2 boundary is χ_{IR}^{-1} .³⁷

Relation between χ_{UV} and χ_{IR} In order to relate χ_{UV} and χ_{IR} , let us relate the two solution bases [21, 2],

$$\rho_\pm = a^{(+)}\phi_\pm + b^{(+)}\varphi_\pm, \quad \psi_\pm = a^{(-)}\phi_\pm + b^{(-)}\varphi_\pm. \quad (106)$$

Here (\pm) implies the powers $\pm 2\nu_k$ in the asymptotic behavior given in (87).³⁸ Thus, plugging (106) into (102), we find³⁹

$$\chi_{\text{IR}}^{-1} = -\frac{a^{(-)} - \chi_{\text{UV}}^{-1}b^{(-)}}{a^{(+)} - \chi_{\text{UV}}^{-1}b^{(+)}}. \quad (107)$$

$a^{(\pm)}, b^{(\pm)}$ can be expanded in ωR (which is very small in the sense of (92)):

$$a^{(\pm)} = a_0^{(\pm)} + (\omega R)a_1^{(\pm)} + \dots, \quad b^{(\pm)} = b_0^{(\pm)} + (\omega R)b_1^{(\pm)} + \dots. \quad (108)$$

Using this expansion together with (107) and (105), we can obtain information about the behaviour of F^\sharp in the UV of the near horizon region.

Semi-holographic action Let us consider a holographic dual of the bulk theory in the inner region, which has the emergent $\text{AdS}_2 \times \mathbb{R}^{d-1}$ metric (93). Let the $\text{AdS}_2 \times \mathbb{R}^{d-1}$ boundary condition be given by a wave-functional of the form (77) at any point in the matching region, which can be interpreted as a cut-off of the $\text{AdS}_2 \times \mathbb{R}^{d-1}$ boundary. Now, we focus on the double trace coupling. As shown above in (105), the renormalized double trace coupling is given by χ_{IR}^{-1} . This means that the wave-functional at the $\text{AdS}_2 \times \mathbb{R}^{d-1}$ boundary in the continuum limit is given by

$$\Psi_{\text{AdS}_2}[\eta_+^\sharp, \bar{\eta}_-^\sharp] := \langle \eta_+^\sharp, \bar{\eta}_-^\sharp | \Psi_{\text{AdS}_2} \rangle = \exp(-\bar{\eta}_-^\sharp F^\sharp \eta_+^\sharp + \dots) = \exp(-\bar{\eta}_-^\sharp \chi_{\text{IR}}^{-1} \eta_+^\sharp + \dots). \quad (109)$$

³⁷Note that since $A^{\text{IR}} = \rho_+^\sharp/\rho_-^\sharp$ trivially vanishes, it is unnecessary to consider counterterms in this discussion.

³⁸ $a^{(\pm)}, b^{(\pm)}$ are analogous to a_\pm, b_\pm in [21]. We use superscripts (\pm) in order to avoid the confusion with the chirality indices, that are written as subscripts \pm in our paper.

³⁹One can see that this relation has the same structure as the relation between G_R and \mathcal{G}_R given in [21], which, in fact, is a special case of (107) because, with the infalling boundary condition at the horizon, χ_{IR} becomes \mathcal{G}_R and χ_{UV} becomes G_R [49].

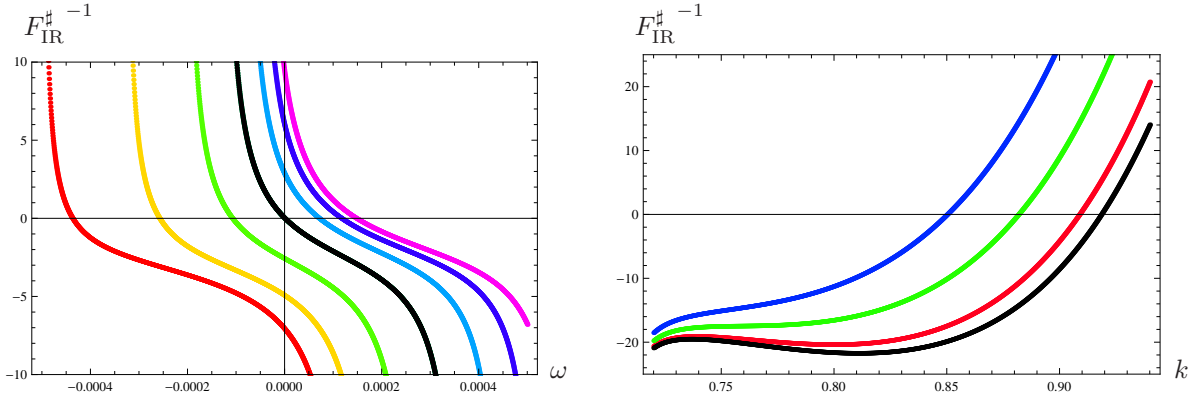


Figure 8: Plots obtained numerically for $m = 0$, $z_* = 1$, $q = 1$, $\mu = \sqrt{3}$. The left panel shows $F_{\text{IR}}^{\sharp-1}$ evaluated near the horizon ($r = \log(z^{-1} - z_*^{-1}) = -10$) as a function of ω for $\chi_{\text{UV}} = 0$ and $k = k_F = 0.91853$ (black), $k = k_F \pm 0.005$, $k = k_F \pm 0.01$, and $k = k_F \pm 0.015$. The right panel shows $F_{\text{IR}}^{\sharp-1}$ evaluated near the horizon ($r = -10$) as a function of k for $\omega = 0$ and $\chi_{\text{UV}} = 0$ (black), $\chi_{\text{UV}} = 0.01$ (red), $\chi_{\text{UV}} = 0.04$ (green), and $\chi_{\text{UV}} = 0.08$ (blue).

Here we have used a twisted basis $\langle \eta_+^{\sharp}, \bar{\eta}_-^{\sharp} |$ which are eigenstates of the twisted field $\widehat{\Phi}^{\sharp} \equiv \widehat{\Phi} := (\widehat{\chi}_+^{\sharp}, \widehat{\chi}_-^{\sharp})^T$ (see (100)). $\bar{\eta}_{-R}^{\sharp} \sim \xi^{-\nu_k} \bar{\eta}_-^{\sharp}$ and $\eta_{+R}^{\sharp} \sim \xi^{-\nu_k} \eta_+^{\sharp}$ are renormalized fields. The wavefunction Ψ_{AdS_2} is to be identified with Ψ of (76), while $(\bar{\eta}_{-R}^{\sharp}, \eta_{+R}^{\sharp})$ is to be identified with $(\bar{\chi}_+, \tilde{\chi}_-)$.

It can be verified numerically that for specific values of $k = k_F$, one has $b_0^{(+)} = 0$. In this case, writing $\vec{k} := k_F \hat{k} + \Delta \vec{k}_{\parallel}$ (where \hat{k} is the unit vector along \vec{k}), $b^{(+)}$ can be expanded in ω and $\Delta \vec{k}_{\parallel}$ as

$$b^{(+)} \sim c_t \omega + c_i \Delta k_{\parallel i} + \dots, \quad (110)$$

where c_t, c_i are numerical constants (see footnote 27). If we choose $\chi_{\text{UV}} = 0$, then we obtain that

$$F^{\sharp} \sim \chi_{\text{IR}}^{-1} = b^{(-)}/b^{(+)} \quad (111)$$

has a single pole at the Fermi momentum k_F for $\omega = 0$. (For generic values of χ_{UV} , the pole shifts relative to k_F .) Therefore, (109) is of the form (77) with \tilde{F} replaced by χ_{IR}^{-1} , which is of the form (80) with $k_*^0 = 0$, $|\vec{k}_*| = k_F$. As mentioned above, $(\bar{\eta}_{+R}^{\sharp}, \eta_{-R}^{\sharp})$ correspond to $(\bar{\chi}_+, \tilde{\chi}_-)$ in (83). Let us now study all this in more detail numerically.

Numerical study The left panels of Figure 8 and Figure 9, show the ω -dependence of $F^{\sharp-1}$ evaluated in the IR for $\chi_{\text{UV}} = 0$ and different values of k . As can be seen, we obtain the characteristic linear behaviour anticipated by (111). The right panels of Figure 8 and Figure 9, show the k -dependence of $F^{\sharp-1}$ evaluated in the IR for $\omega = 0$ and different values of χ_{UV} . Making χ_{UV} non-zero has the effect of shifting the location of the pole of F^{\sharp} . The left panel of Figure 10 shows that the imaginary part of the AdS_4 retarded Greens function has a peak for $k = k_F$. This gives an independent verification of the location of k_F , which agrees with the plots of Figure 8 for $\chi_{\text{UV}} = 0$.

The right panel of Figure 10 shows the flow of $F^{\sharp-1}$ as a function of the radial coordinate, and $\chi_{\text{UV}} = 0$. $F^{\sharp-1}$ starts from the $F^{\sharp-1} = b/d$ in the UV (corresponding to $F = \infty$ or alternative quantization). For $k = k_F$ and $\omega = 0$, the flow approaches $F^{\sharp-1} = 0$ or the alternative quantization in the near horizon region. Non-zero values of ω eventually drive the flow away from $F^{\sharp-1} = 0$ in the deep IR. As k is made to differ from k_F by a small amount, the flow only stays near the alternative fixed point of $\text{AdS}_2 \times \mathbb{R}^{d-1}$ for a while, until it starts flowing towards the attractive standard fixed point at $F^{\sharp-1} = \infty$. This shifts the value of ω for which $F^{\sharp-1}$ hits a zero.

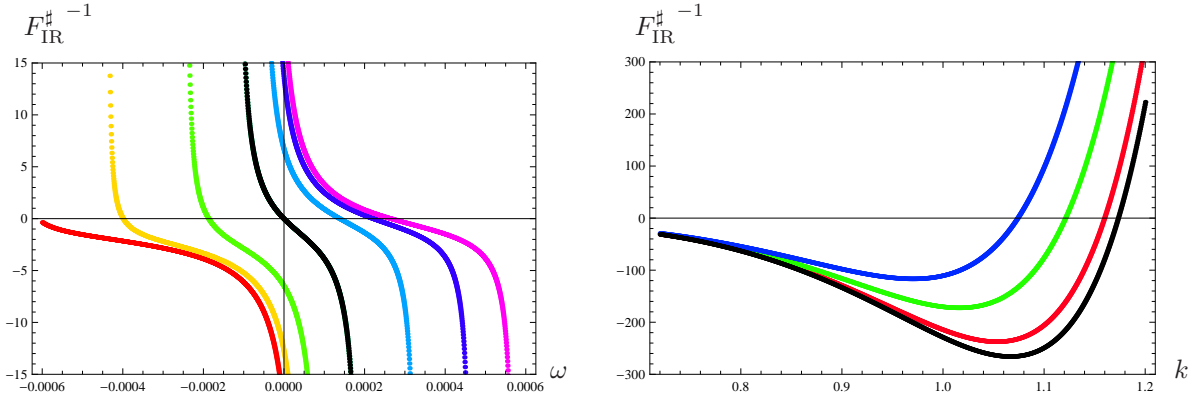


Figure 9: Plots obtained numerically for $m = 0.2$, $z_* = 1$, $q = 1$, $\mu = \sqrt{3}$. The left panel shows $F_{\text{IR}}^{\sharp-1}$ evaluated near the horizon ($r = \log(z^{-1} - z_*^{-1}) = -10$) as a function of ω for $\chi_{\text{UV}} = 0$ and $k = k_F = 1.17309$ (black), $k = k_F \pm 0.001$, $k = k_F \pm 0.002$, and $k = k_F \pm 0.003$. The right panel shows $F_{\text{IR}}^{\sharp-1}$ evaluated near the horizon ($r = -10$) as a function of k for $\omega = 0$ and $\chi_{\text{UV}} = 0$ (black), $\chi_{\text{UV}} = 0.01$ (red), $\chi_{\text{UV}} = 0.04$ (green), and $\chi_{\text{UV}} = 0.08$ (blue).

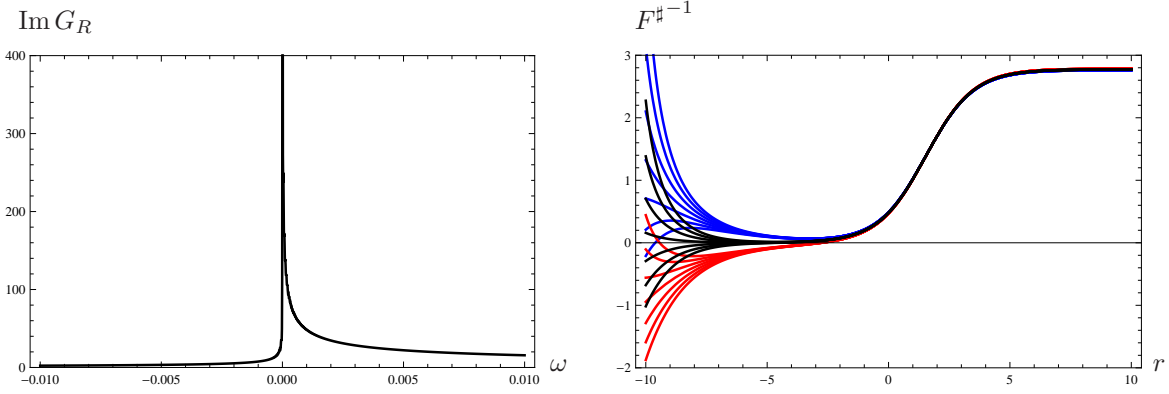


Figure 10: Plots obtained numerically for $m = 0$, $z_* = 1$, $q = 1$, $\mu = \sqrt{3}$. The left panel shows the imaginary part of the UV retarded Greens function G_R for $k = k_F = 0.91853$ as a function of ω . The right panel shows $F_{\text{IR}}^{\sharp-1}$ as a function of $r = \log(z^{-1} - z_*^{-1})$ for $\chi_{\text{UV}} = 0$ and different values of k and ω . The black lines are for $k = k_F$, while the red and blue lines are for $k = k_F - 0.002$ and $k = k_F + 0.002$, respectively. The values of ω fall between $\omega = \pm 4.5 \times 10^{-5}$.

7 Field theory summary

In this section, we will transcribe our findings in the previous sections in field theoretic terms.

7.1 Scalar operators

We will denote the matrix field theory, implicit on the RHS of (4), as follows

$$Z[\epsilon] = \int \mathcal{D}M_\epsilon \exp \left[S[M] + \int_k h(k, \epsilon) O(-k) + \int_k \frac{1}{2} g(k, \epsilon) O(k) O(-k) + C(\epsilon) \right], \quad (112)$$

where M is a d -dimensional matrix field, and O is a certain single trace scalar operator. Typically, a perturbation by O would mix with other operators under Wilsonian RG; the generalization to multiple scalar operators (dual to multiple bulk scalar fields) is straightforward if all bulk scalars are given by quadratic actions such as in footnote 5. However, for simplicity, we will assume a situation in which the operator O here does not mix with others under RG flow, and under $\epsilon \rightarrow \epsilon'$, the partition function Z can be kept invariant by appropriate change of the couplings h, g ; we have also kept an ϵ -dependent zero-point energy $C(\epsilon)$.

The field theory couplings $g(k)$ and $h(k)$ (in the standard quantization) are related to $f(k), J(k)$ of (3) as

$$g(k) = 1/f(k), \quad h(k) = J(k)/f(k). \quad (113)$$

Using (6) we get the beta-functions for g, h .

Example 1 We will consider $S[M] \equiv S_1[M]$ which admits a pure AdS_{d+1} dual. The beta-functions of h, g (for the $k = 0$ mode) are, in this case, given by (recalling $\epsilon \partial/\partial \epsilon = -\Lambda \partial/\partial \Lambda$)

$$-\beta_g = \dot{g} = 1 - dg - m^2 g^2, \quad -\beta_h = \dot{h} = -(d + m^2 g)h. \quad (114)$$

The double trace flow has two fixed points:

(a) $g_* = 1/\Delta_+$; this CFT corresponds to the standard CFT. *Proof:* Recall that for a CFT deformed by a coupling $\int g \mathbb{O}$, the scaling dimension γ of \mathbb{O} , also denoted as $[\mathbb{O}]$, is related to the β -function of g as follows

$$\gamma = [\mathbb{O}] = \partial \beta_g / \partial g. \quad (115)$$

For the double trace coupling under discussion, the fixed point value is g_* , hence the deformation from the CFT is $\int_k \delta g \mathbb{O}$, where $\delta g = g - g_*$ and $\mathbb{O} = O(k)O(-k)$. From (114), $\beta_{\delta g} = 2\nu \delta g + m^2 \delta g^2$, hence $\gamma = 2\nu + m^2 \delta g$. Thus, at the fixed point $[O(k)O(-k)] \equiv \gamma = 2\nu$; hence $[O(x)^2] = d + 2\nu = 2\Delta_+$. By large N factorization, $[O(x)] = \sqrt{[O(x)^2]} = \Delta_+$, implying standard quantization.

(b) $g_* = 1/\Delta_-$: alternative quantization, in the Klebanov-Witten window $0 \leq \nu \leq 1$ [24]. The proof is similar to the above case. In this case

$$\beta_{\delta g} = -2\nu \delta g + m^2 \delta g^2. \quad (116)$$

Note that outside the KW window, the fixed point $g_* = 1/\Delta_-$, although it formally exists, does not represent a valid quantization. See Table 1 for a summary of the two fixed points.

Comparison with a toy model with a Wilson-Fisher fixed point In [17] we consider a $d = 4 - \epsilon$ dimensional free matrix field theory $S[M] = \int d^d x \frac{1}{2} \text{Tr}(\partial M)^2$, perturbed by $\int h' O(x) + \frac{1}{2} g' O(x)^2$, with $O(x) = \text{Tr} M(x)^2$. We find

$$\beta_{g'} = -\epsilon g' + \alpha^2 g'^2 + \dots, \quad (117)$$

where α^2 is a numerical constant⁴⁰. The free field UV fixed point at $g' = 0$ is superficially comparable with the UV CFT for alternative quantization described by the beta-function (116). The point to stress here is that the beta-function (116), obtained by holographic means, is exact even for strong coupling.

Example 2 The field theory $S[M] \equiv S_2[M]$ has a global $U(1)$ symmetry, and admits a charged black hole (9) dual. The existence of a normalizable δg_{tt} , at least for small $z = \epsilon$, can be interpreted as an expectation value $\langle T_{tt} \rangle$. The running of δg_{tt} , interpreted as a flow of $\langle T_{tt} \rangle$ ⁴¹, can be read off from the exact dependence of g_{tt} on z as given by $H(z)$ in (9). With this interpretation of the background geometry in mind, we again perturb the theory as in (112) and obtain the beta-functions for g, h by substituting (113) in (5), (6) and (10).

Along with $\langle T_{tt} \rangle$ there is also a $\langle J_t \rangle$ caused by the presence of the $U(1)$ gauge field A_t in the dual geometry. The fact that the dual charged black hole is extremal means that $\langle J_t \rangle$ is determined by $\langle T_{tt} \rangle$ and need not be considered separately. Extremality of the dual also implies the existence of 2 additional fixed points as $z = \epsilon \rightarrow z_*$ which are the AdS_2 analogs of the 2 fixed points discussed in Example 1: the non-zero values of $\langle T_{tt} \rangle, \langle J_t \rangle$ are interpreted in terms of a new d -dimensional CFT on $\mathbb{R}^1 \times \mathbb{R}^{d-1}$. The RG flows and the four fixed points are described in Section 2 and in Figure 1.

⁴⁰We note that in this field theory toy model, the other marginal operator $\text{Tr} M(x)^4$, if absent initially, does not get generated by the flows of the operators $O(x), O(x)^2$.

⁴¹Similar interpretations have been used by, e.g. [50, 51], for two-point functions of currents or stress tensors in the context of cut-off dependence of transport coefficients.

7.2 Fermionic operators

The difference with (112) is that this time we perturb the theory by a (single-trace) fermionic operator O_a (where a is a Dirac index)

$$Z[\epsilon] = \int \mathcal{D}M_\epsilon \exp \left[S(M, \epsilon) + \int_k \bar{h}_a(k, \epsilon) O_a(k) + \int_k \bar{O}_a(k) h_a(k) + \int_k \frac{1}{2} \bar{O}_a(k) g_{ab}(k, \epsilon) O_b(k) \right]. \quad (118)$$

Comparing with (33) we find that in the spinorial basis we use

$$O_a(k) g_{ab}(k, \epsilon) O_b(k) = \bar{O}_- F(\epsilon) O_+, \quad \bar{h}_a(k, \epsilon) O_a(k) = \bar{O}_- J_-(\epsilon), \quad \bar{O}_a(k) h_a(k) = \bar{J}_+(\epsilon) O_+.$$

In other words, double trace and single trace couplings are given directly in terms of F, J , whose running is given by (37).

Example 1 As in the bosonic case, the field theory $S_1[M]$ is assumed to admit a pure AdS_{d+1} dual. The beta-function for the double trace coupling is given by (49), or alternatively by (51). In terms of the latter, the zero-momentum coupling is f_0 which, according to the flow equations (53), (55), flows to the IR fixed point $f_{0,*} = 0$ from the UV fixed point $f_{0,*} = \infty$.

The nature of the fixed points can be found as in the bosonic case above. Thus, near $f_{0,*} = 0$, the double trace coupling is $f_0 \bar{O}_- O_+$, hence by (115), we have $[\bar{O}_-(k) O_+(k)] = \partial \beta_{f_0} / \partial f_0 = 2m$ (recall $\epsilon \partial f_0 / \partial \epsilon = -\Lambda \partial f_0 / \partial \Lambda \equiv \beta_{f_0}$). In position space we obtain $[\bar{O}_-(x) O_+(x)] = d + 2m$. Using large N factorization, we obtain $\gamma_{\bar{O}_-} = \gamma_{O_+} = d/2 + m \equiv \Delta_+$, which says that $f_{0,*} = 0$ corresponds to standard quantization. The fact that $[\bar{O}_- O_+] > d$ implies that it is an irrelevant operator, hence the standard fixed point is an IR fixed point; this agrees with the attractive nature of the fixed point, as found in (53).

The nature of the fixed point $f_{0,*} = \infty$, or equivalently $g_{0,*} = 0$ can be similarly found. The coupling now is $g_0 \bar{O}_+ O_-$ which, using (115), gives $[\bar{O}_+ O_-] = -2m$, or $[\bar{O}_+(x)] = [O_-(x)] = d/2 - m$, which signifies alternative quantization. The fact that $\bar{O}_+ O_-$ turns out to be a relevant operator accords with the fact that the alternative fixed point, by (55), is repulsive, and is hence a UV fixed point.

Example 2 As in the bosonic case, the field theory $S_2[M]$ is assumed to admit an extremal charged black hole dual, with scale-dependent $\langle T_{tt} \rangle, \langle J_t \rangle$. The beta-function for the double trace coupling is given by (64). As in the bosonic case, the double trace flow, together with the flow of $\langle T_{tt} \rangle$, connect the two fixed points at very high energies (the boundary AdS_{d+1} region) to two fixed points in the IR (the near-horizon AdS_2 region), see Figure 7.

7.3 Emergence of new (bosonic/fermionic) degrees of freedom

As discussed in Section 6, the double trace coupling $g(k)$ in Example 2 above can develop a pole singularity at the Fermi surface⁴² This implies emergence of new light degrees of freedom in the field theory. We can trace their origin as follows.

Low energy theory \rightarrow random (dynamical) source Equations (4) and (31) can be interpreted as follows: holographic version of Wilsonian RG can be regarded as appearance of new dynamical degrees of freedom, in the form of

⁴² For simple perturbations around standard free field theories, operators with zero momentum are more relevant than those with finite momenta (higher derivatives); the fact that the double trace coupling at a finite momentum blows up is due to the black hole background which represents strong coupling physics.

random (fluctuating) sources ⁴³

$$Z[\epsilon_0] = \int DM_{\epsilon_0} \exp \left(S_0[M] + \int J_0 O[M] \right) = Z[\epsilon] = \int DM_{\epsilon} DJ \exp(S[J]) \exp \left(S[M] + \int JO[M] \right). \quad (119)$$

Generically we can integrate out the random sources, yielding a low energy effective action in terms of the original field variables. However, in special situations (e.g. on the Fermi surface, as discussed above in Section 6.1), $J(k)$ becomes ‘massless’ and can be identified as an emergent degree of freedom (bosonic or fermionic, depending on whether O is bosonic or fermionic), which must be kept in the dynamics to keep a local description of the theory; see (83) as an example.

Comparison with Gross-Neveu model and QCD The essence of holographic Wilsonian RG, as captured by (119), can be compared with the Nambu–Jona-Lasinio (NJL) approach to QCD (see, e.g. [30, 31], or in a simpler setting, with the Gross-Neveu model [29]. By introducing a dynamical source term the partition function of the latter model can be written as

$$Z = \int D\bar{\psi} D\psi \exp \left[\int d^2x \left(\bar{\psi} \not{\partial} \psi + \frac{g^2}{2} (\bar{\psi} \psi)^2 \right) \right] = \int DJ D\bar{\psi} D\psi \exp \left[- \int \frac{J^2}{2g^2} \right] \exp \left[\int (\bar{\psi} \not{\partial} \psi + J \bar{\psi} \psi) \right]. \quad (120)$$

The fermions transform in the fundamental representation of $U(N_f)$. In the second expression for the partition function, integrating out the fermions gives an effective action $S_{\text{eff}}[J(k)]$. The source field $J(x) \sim \bar{\psi}(x)\psi(x)$ is a rough analog of mesons in the QCD problem [30, 31]. The last expression of (120) is analogous to the last expression of (119); the fluctuating source of (119) (for bosonic $J(k)$) can be identified with the ‘meson’ fields. The effective action S_{eff} can be computed exactly at large N_f , and exhibits a symmetry breaking potential.

A massless $J(k)$ field, in such contexts, can emerge as follows: suppose that the fermions in (120) also transform under a second flavour group $U(N'_f)$, under which the mesons transform nontrivially. In this case $S_{\text{eff}}[J(k)]$ is invariant under $U(N'_f)$; in the symmetry broken phase, some of the source fields $J(k)$ can be identified as Goldstone bosons (pions). See [4] for discussions along these lines. It would be interesting to find an analog of the above picture for emergent massless fermions; see a related discussion in [26] in the context of the ‘FL*’ phase.

8 Concluding remarks

In this work, we have described a framework of how to carry out the program of holographic Wilsonian RG (hWRG) with fermions. More details will appear in [17]. As mentioned in the Introduction, we found a precise meaning of the existence of a Fermi surface in terms of singular effective double trace coupling in the AdS_2 region; we also found a dynamical origin of the semi-holographic origin of Faulkner and Polchinski.

We end with a few open questions and generalizations:

- It is of obvious interest to identify specific strongly coupled field theories which have holographic duals of the kind described in this paper. Work in this direction is in progress [17] (see the discussion around (117) for an example).
- The random source interpretation of hWRG (119) gives an insight into Wilsonian RG of strongly coupled field theories [52, 53]. Random source models have an extensive literature in statistical mechanics; an example is the so-called compressible Ising model with random couplings on the links (see [54], e.g.). It would be interesting to see if these models too can have emergent degrees of freedom as we found here.

⁴³The field theory variables are generically denoted as a matrix field $M(x)$, as below (82). $O[M]$ is a single trace operator; the Equation (119) is valid if this operator does not mix with others under RG. More generally, one should include all single trace operators $O_n[M]$ and the corresponding sources J_n .

- The formalism developed in this paper is applicable to any asymptotically AdS background, especially those which are translationally invariant in the boundary directions. The discussions of RG flows can be easily generalized to the case of non-extremal black hole where the theory at IR is massive; for nearly extremal black holes, our discussion on semi-holography generalizes rather easily, including the appearance of an emergent fermion at the AdS_2 boundary. Our formalism can also be applied to Lifshitz geometries [55, 56] with different dynamical exponents in the UV and IR [17]. It would also be interesting to see if we can apply hWRG in reverse [1]: starting out from a IR conformal field theory and flowing towards the UV. The semi-holographic viewpoint does not require the existence of a UV fixed point; in particular the UV theory is allowed to be a lattice theory. It would be obviously important to complete such a picture of hWRG.
- In addition to the flow of couplings, it is important to compute the flow of correlation functions; in particular, it would be interesting to understand RG flows of transport coefficients using hWRG (see, *e.g.* [50, 51, 57]).

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A Notations

A.1 Miscellaneous notations

coordinates and momenta

$$x^i := (x^1, x^2, \dots, x^{d-1}), \quad x^\mu := (t, x^i), \quad x^M := (z, x^\mu),$$

$$k_i := (k_1, k_2, \dots, k_{d-1}), \quad k_\mu := (\varepsilon\omega, k_i), \quad k^2 := g^{\mu\nu} k_\mu k_\nu, \quad \mathbf{k}^2 := \eta^{\hat{\mu}\hat{\nu}} k_\mu k_\nu = \varepsilon\omega^2 + k_i k_i,$$

where $\varepsilon = 1(-1)$ for Euclidean (Lorentzian) metric.

metric, covariant derivative and spin connections

$$ds^2 = g_{MN} dx^M dx^N = \eta_{\hat{a}\hat{b}} \theta^{\hat{a}} \theta^{\hat{b}}, \quad \eta_{\hat{a}\hat{b}} := \text{diag}\{1, \varepsilon, 1, \dots, 1\}, \quad g := \det(g_{MN}),$$

$$D_M \psi := \partial_M \psi + \frac{1}{4} \omega_{M\hat{a}\hat{b}} \Gamma^{\hat{a}\hat{b}} \psi - iq A_M \psi, \quad \overline{D_M \psi} := \partial_M \overline{\psi} - \frac{1}{4} \omega_{M\hat{a}\hat{b}} \overline{\psi} \Gamma^{\hat{a}\hat{b}} + iq A_M \overline{\psi},$$

$$\omega^{\hat{a}\hat{b}} = \omega_M^{\hat{a}\hat{b}} dx^M, \quad d\theta^{\hat{a}} + \omega_{\hat{b}}^{\hat{a}} \wedge \theta^{\hat{b}} = 0. \quad (121)$$

Fourier transformations

$$\psi(z, x) := \int \frac{d^d k}{(2\pi)^d} e^{ikx} \psi(z, k), \quad \overline{\psi}(z, x) := \int \frac{d^d k}{(2\pi)^d} e^{-ikx} \overline{\psi}(z, k), \quad kx := k_\mu x^\mu = \varepsilon\omega t + k_i x^i.$$

A.2 Dirac action

We give a general notation which applies to both Euclidean and Lorentz signatures. The Dirac action is

$$S := \mathcal{N} \int dz d^d x \sqrt{\varepsilon g} (\overline{\psi} \Gamma^M D_M \psi - m \overline{\psi} \psi).$$

where $g := \det(g_{MN})$.⁴⁴ A numerical factor \mathcal{N} is chosen so that the path integral representation is $\int [d\psi] e^{-S}$ both for Euclidean and Lorentzian signature. If we compare with the Dirac actions given in [49], our \mathcal{N} is related to \mathcal{N} in [49] (which we write as \mathcal{N}_{IL}), by $\mathcal{N} = \mathcal{N}_{\text{IL}}$ for the Lorentzian metric, and $\mathcal{N} = -\mathcal{N}_{\text{IL}}$ for the Euclidean metric. In this paper, we set $\mathcal{N} = 1$ just for notational simplicity⁴⁵ except the next section. In order to revive \mathcal{N} , we just have to replace κ^{-2} by $\kappa^{-2}\mathcal{N}$.

B Generalized coherent states

In this section, we will investigate the state space of a quantized Dirac fermion in the general background. For simplicity, we will omit spinor indices and momenta variables and momentum integral symbols. This is equivalent to picking up a Hilbert space with one specific momentum and one specific spinor component. This is consistent because the anti-commutation relations we will use have no mixing between different momenta and spinor components. The starting point is the anti-commutation relation

$$\{\widehat{\chi}_{\pm}, \pm \mathcal{N}_d \widehat{\chi}_{\pm}\} = 1, \quad \mathcal{N}_d := \kappa^{-2} (2\pi)^{-d} \mathcal{N}. \quad (122)$$

First, we construct a Fock space with regarding $\widehat{\chi}_{\pm}$ as annihilation operators and $\pm \mathcal{N}_d \widehat{\chi}_{\pm}$ as creation operators starting from a ground state $|0, 0\rangle$, which is annihilated by $\widehat{\chi}_{\pm}$. The basis of the Fock space is $\{|0, 0\rangle, |1, 0\rangle, |0, 1\rangle, |1, 1\rangle\}$ with

$$|1, 0\rangle := \mathcal{N}_d \widehat{\chi}_+ |0, 0\rangle, \quad |0, 1\rangle := -\mathcal{N}_d \widehat{\chi}_- |0, 0\rangle, \quad |1, 1\rangle := \mathcal{N}_d \widehat{\chi}_+ (-\mathcal{N}_d \widehat{\chi}_-) |0, 0\rangle. \quad (123)$$

Next, the dual basis is defined so that

$$\langle\langle m, n | p, q \rangle\rangle = \delta_{mp} \delta_{nq}. \quad (124)$$

The dual basis is constructed from the dual ground state with $\langle\langle 0, 0 | 0, 0 \rangle\rangle = 1$,

$$\langle\langle 1, 0 | := \langle\langle 0, 0 | \widehat{\chi}_+, \quad \langle\langle 0, 1 | := \langle\langle 0, 0 | \widehat{\chi}_-, \quad \langle\langle 1, 1 | := \langle\langle 0, 0 | \widehat{\chi}_- \widehat{\chi}_+. \quad (125)$$

We now define “generalized coherent states”, which satisfy

$$\begin{aligned} \widehat{\chi}_+ |\bar{u}, v\rangle &= \bar{u} |\bar{u}, v\rangle, & \widehat{\chi}_- |\bar{u}, v\rangle &= v |\bar{u}, v\rangle, & \widehat{\chi}_+ |u, \bar{v}\rangle &= u |u, \bar{v}\rangle, & \widehat{\chi}_- |u, \bar{v}\rangle &= \bar{v} |u, \bar{v}\rangle, \\ \langle u, \bar{v} | \widehat{\chi}_+ &= \langle u, \bar{v} | u, & \langle u, \bar{v} | \widehat{\chi}_- &= \langle u, \bar{v} | v, & \langle \bar{u}, v | \widehat{\chi}_+ &= \langle \bar{u}, v | \bar{u}, & \langle \bar{u}, v | \widehat{\chi}_- &= \langle \bar{u}, v | v. \end{aligned} \quad (126)$$

They are explicitly constructed as follows⁴⁶

$$\begin{aligned} |\bar{u}, v\rangle &:= e^{\mathcal{N}_d (-\bar{u} \widehat{\chi}_+ - \widehat{\chi}_- v)} |1, 0\rangle, & |u, \bar{v}\rangle &:= e^{\mathcal{N}_d (\widehat{\chi}_+ u + \bar{v} \widehat{\chi}_-)} |0, 1\rangle, \\ \langle u, \bar{v} | &:= \langle\langle 1, 0 | e^{\mathcal{N}_d (-\widehat{\chi}_+ u - \bar{v} \widehat{\chi}_-)}, & \langle \bar{u}, v | &:= \langle\langle 0, 1 | e^{\mathcal{N}_d (\bar{u} \widehat{\chi}_+ + \widehat{\chi}_- v)}. \end{aligned} \quad (128)$$

Inner products of generalized coherent states are

$$\begin{aligned} \langle p, \bar{q} | \bar{u}, v\rangle &= e^{\mathcal{N}_d (-\bar{u} p - \bar{q} v)}, & \langle \bar{p}, q | u, \bar{v}\rangle &= e^{\mathcal{N}_d (\bar{p} u + \bar{v} q)}, \\ \langle p, \bar{q} | u, \bar{v}\rangle &= \mathcal{N}_d \delta(p - u) \delta(\bar{q} - \bar{v}), & \langle \bar{p}, q | \bar{u}, v\rangle &= \mathcal{N}_d \delta(\bar{p} - \bar{u}) \delta(q - v). \end{aligned} \quad (129)$$

Generalized coherent states satisfy the completeness relations,

$$\widehat{1} = \frac{1}{\mathcal{N}_d^2} \int d\bar{u} dv du d\bar{v} |\bar{u}, v\rangle \langle u, \bar{v}| e^{\mathcal{N}_d (\bar{u} u + \bar{v} v)} = \frac{1}{\mathcal{N}_d^2} \int du d\bar{v} d\bar{u} dv |u, \bar{v}\rangle \langle \bar{u}, v| e^{\mathcal{N}_d (-\bar{u} u - \bar{v} v)}, \quad (130)$$

$$\widehat{1} = \frac{1}{\mathcal{N}_d} \int dv d\bar{u} |\bar{u}, v\rangle \langle \bar{u}, v| = \frac{1}{\mathcal{N}_d} \int d\bar{v} du |u, \bar{v}\rangle \langle u, \bar{v}|. \quad (131)$$

⁴⁴ $g_{tt} > 0$ for Euclidean case and $g_{tt} < 0$ for Lorentzian case

⁴⁵ In fact, \mathcal{N}_{IL} cannot be arbitrary; it is constrained by the unitarity of the bulk theory.

⁴⁶ We often see a form like $\mathcal{N}_d (\bar{a} b + \bar{c} d)$ in the exponent. In the field theory, spinor indices and momentum dependence are recovered as follows

$$\mathcal{N}_d (\bar{a} b + \bar{c} d) \longrightarrow \mathcal{N}_d \int d^d k (\bar{a}(k) b(k) + \bar{c}(k) d(k)) = \mathcal{N} \int_k (\bar{a}(k) b(k) + \bar{c}(k) d(k)). \quad (127)$$

Proof of the replacement rule of the Schrödinger equation Here we give a derivation of the replacement rules (36) of the Schrödinger equation. From the inner product above, we find

$$\begin{aligned} & \langle \eta_+, \bar{\eta}_- | \hat{H} | \bar{\chi}_+, \chi_- \rangle \\ &= \hat{H}(\hat{\chi}_+ \rightarrow \bar{\chi}_+, \hat{\chi}_- \rightarrow \chi_-, \hat{\chi}_+ \rightarrow \eta_+, \hat{\chi}_- \rightarrow \bar{\eta}_-) e^{\mathcal{N}_d(-\bar{\chi}_+ \eta_+ - \bar{\eta}_- \chi_-)} \\ &= \hat{H}\left(\hat{\chi}_+ \rightarrow \frac{1}{\mathcal{N}_d} \frac{\delta}{\delta \eta_+}, \hat{\chi}_- \rightarrow -\frac{1}{\mathcal{N}_d} \frac{\delta}{\delta \bar{\eta}_-}, \hat{\chi}_+ \rightarrow \eta_+, \hat{\chi}_- \rightarrow \bar{\eta}_-\right) e^{\mathcal{N}_d(-\bar{\chi}_+ \eta_+ - \bar{\eta}_- \chi_-)}. \end{aligned}$$

One can see that the derivation is independent of the details of the Hamiltonian and the replacement rule applies to Hamiltonians with interactions. Note that one also needs to be careful about the ordering of operators in the Hamiltonian, as can be seen from, for instance, the mass terms of (17).

C Holographic Wilsonian RG for Dirac fermions in a rotationally symmetric background

We consider Dirac equation on a rotationally symmetric background with $A_i = 0$ and using a specific representation of Dirac matrices following [21]. We find that the flow equations (37) simplify considerably and construct their solutions in terms of classical solutions of the Dirac equations. In this section, we will use the Lorentzian signature.

C.1 Dirac equation and classical solutions

The action (14) in the momentum space becomes

$$S = \int dz \int_k \left(\bar{\Phi} \Gamma^{\hat{z}} \partial_z \Phi + \sqrt{g_{zz}} [\bar{\Phi} (-i \Gamma^t u_t + i \Gamma^i k_i) \Phi - m \bar{\Phi} \Phi] \right). \quad (132)$$

with $u_t := \omega + q A_t$. The equations of motion can be written as

$$\begin{aligned} D\Phi &= 0, \quad D := \Gamma^{\hat{z}} \partial_z - m \sqrt{g_{zz}} - i \sqrt{-g_{zz} g^{tt}} \Gamma^{\hat{t}} u_t + i \sqrt{g_{zz} g^{ii}} \Gamma^{\hat{i}} k_i, \\ \bar{\Phi} \bar{D} &= 0, \quad \bar{D} := \Gamma^{\hat{z}} \partial_z + m \sqrt{g_{zz}} + i \sqrt{-g_{zz} g^{tt}} \Gamma^{\hat{t}} u_t - i \sqrt{g_{zz} g^{ii}} \Gamma^{\hat{i}} k_i. \end{aligned} \quad (133)$$

Without loss of generality, we can choose a special k due to the rotational symmetry in the spatial directions

$$k := k_1, \quad \hat{k}_1 := 1, \quad \hat{k}_2, \dots, \hat{k}_{d-1} := 0. \quad (134)$$

For simplicity, we consider the case $d = 3$ (the extension to higher dimensions is given in Appendix C.3). The Dirac operator commutes with following two projectors

$$[D, \Pi_\alpha] = [\bar{D}, \Pi_\alpha] = 0, \quad \Pi_\alpha \Pi_\beta = \delta_{\alpha\beta} \Pi_\beta, \quad \Pi_\alpha := \frac{1 - (-)^\alpha \hat{k}_i \Gamma^{\hat{i}} \Gamma^{\hat{z}} \Gamma^{\hat{t}}}{2}, \quad \hat{k}_i \hat{k}_i = 1, \quad \alpha = 1, 2. \quad (135)$$

We adopt such a representation of the Dirac matrices that the Dirac operator is diagonal in terms of the projections

$$\begin{aligned} \Gamma^{\hat{z}} &= \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad \Gamma^{\hat{t}} = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}, \quad \Gamma^{\hat{1}} = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad \Gamma^{\hat{2}} = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \\ \Pi_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Pi_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \quad \bar{\Phi} = (\bar{\Phi}_1, \bar{\Phi}_2). \end{aligned} \quad (136)$$

Then, the action and the equations of motion become

$$\begin{aligned} S &= \sum_\alpha \int dz \int_k (\bar{\chi}_\alpha^+ \partial_z \chi_\alpha^+ - \bar{\chi}_\alpha^- \partial_z \chi_\alpha^- + \sqrt{g_{zz}} [-\bar{\chi}_\alpha^+ T_\alpha^- \chi_\alpha^- + \bar{\chi}_\alpha^- T_\alpha^+ \chi_\alpha^+ - m \bar{\chi}_\alpha^+ \chi_\alpha^+ - m \bar{\chi}_\alpha^- \chi_\alpha^-]) , \\ (\sqrt{g^{zz}} \partial_z \mp m) \chi_\alpha^\pm &= T_\alpha^\mp \chi_\alpha^\mp, \quad (\sqrt{g^{zz}} \partial_z \pm m) \bar{\chi}_\alpha^\pm = T_\alpha^\pm \bar{\chi}_\alpha^\mp, \\ \Phi_\alpha &= \begin{pmatrix} \chi_\alpha^+ \\ \chi_\alpha^- \end{pmatrix}, \quad \bar{\Phi}_\alpha = (\bar{\chi}_\alpha^+, \bar{\chi}_\alpha^-), \quad T_\alpha^\pm := \pm \sqrt{-g^{tt}} u_t - (-)^\alpha \sqrt{g^{11}} k. \end{aligned}$$

The indices \pm imply the eigenvalues ± 1 of σ_3 , which are actually equal to the $\Gamma^{\hat{z}}$ -chirality. The radial Hamiltonian and the anticommutation relations are defined by

$$\begin{aligned}\hat{H} &:= \int \frac{d^d k}{(2\pi)^d} \sqrt{g_{zz}} \left[-\bar{\chi}_\alpha^+ T_\alpha^- \chi_\alpha^- + \bar{\chi}_\alpha^- T_\alpha^+ \chi_\alpha^+ + m \chi_\alpha^+ \bar{\chi}_\alpha^+ - m \bar{\chi}_\alpha^- \chi_\alpha^- \right], \\ \{\chi_\alpha^\pm, \bar{\chi}_\beta^\pm\} &= \pm \kappa^2 (2\pi)^d \delta_{\alpha\beta},\end{aligned}\tag{137}$$

where the summation symbol over α is omitted in the radial Hamiltonian. We can prove that the radial Hamiltonian and the anticommutation relations reproduce the classical equations of motion.

The component expressions of the equations of motion are given by

$$(\sqrt{g^{zz}} \partial_z \mp m) \Phi_\alpha^\pm = T_\alpha^\mp \Phi_\alpha^\mp, \quad (\sqrt{g^{zz}} \partial_z \pm m) \bar{\Phi}_\alpha^\pm = T_\alpha^\pm \bar{\Phi}_\alpha^\mp.\tag{138}$$

One can see from the equations of motion that Φ_α^\pm and $\bar{\Phi}_\alpha^\mp$ satisfy the same equations. From the coupled first order equations, one can get a decoupled second order equation for each component

$$\begin{aligned}(\sqrt{g^{zz}} \partial_z \pm m - \sqrt{g^{zz}} \partial_z \log T_\alpha^\mp) (\sqrt{g^{zz}} \partial_z \mp m) \Phi_\alpha^\pm - (g^{ii} k^2 + g^{tt} u_t^2) \Phi_\alpha^\pm &= 0, \\ (\sqrt{g^{zz}} \partial_z \mp m - \sqrt{g^{zz}} \partial_z \log T_\alpha^\pm) (\sqrt{g^{zz}} \partial_z \pm m) \bar{\Phi}_\alpha^\pm - (g^{ii} k^2 + g^{tt} u_t^2) \bar{\Phi}_\alpha^\pm &= 0.\end{aligned}\tag{139}$$

Let us introduce a basis of solutions to (138) and express any solution to (138) in terms of the basis,

$$\Phi_\alpha^\pm = A_\alpha \phi_\alpha^\pm + B_\alpha \varphi_\alpha^\pm, \quad \bar{\Phi}_\alpha^\pm = \bar{A}_\alpha \bar{\phi}_\alpha^\pm + \bar{B}_\alpha \bar{\varphi}_\alpha^\pm,\tag{140}$$

where (+)-component and (-)-component are related by (138). The numerical coefficients $(A_\alpha, B_\alpha, \bar{A}_\alpha, \bar{B}_\alpha)$ are fixed by boundary conditions. Examples of the solution bases are given in Section 6.

C.1.1 AdS_{d+1}

The solutions to (139) are given by $(\nu_\pm = m \pm \frac{1}{2})$

$$\begin{aligned}\Phi^\pm &= \sqrt{z} (a_\pm(k_\mu) I_{-\nu_\mp}(zK) + b_\pm(k_\mu) I_{\nu_\mp}(zK)), \\ \bar{\Phi}^\pm &= \sqrt{z} (\bar{a}_\pm(k_\mu) I_{-\nu_\pm}(zK) + \bar{b}_\pm(k_\mu) I_{\nu_\pm}(zK)).\end{aligned}\tag{141}$$

where $K^2 \equiv k^2 - (\omega + \mu_q)^2 \neq 0, \omega \neq 0$, and we have suppressed the α -index. The requirement that these solutions (141) satisfy the Dirac equation Eq. (138), relates the integration constants as

$$\begin{aligned}a_-(k_\mu) &= \frac{k + \omega + \mu_q}{K} a_+(k_\mu), \quad b_-(k_\mu) = \frac{k + \omega + \mu_q}{K} b_+(k_\mu), \\ \bar{a}_+(k_\mu) &= \frac{k + \omega + \mu_q}{K} \bar{a}_-(k_\mu), \quad \bar{b}_+(k_\mu) = \frac{k + \omega + \mu_q}{K} \bar{b}_-(k_\mu).\end{aligned}\tag{142}$$

For reference, we give the asymptotic behavior ($z \sim 0$) of the Bessel functions,

$$\sqrt{z} I_{-\nu_+}(zK) \sim z^{-m}, \quad \sqrt{z} I_{-\nu_-}(zK) \sim z^{-m+1}, \quad \sqrt{z} I_{\nu_+}(zK) \sim z^{m+1}, \quad \sqrt{z} I_{\nu_-}(zK) \sim z^m.\tag{143}$$

C.1.2 Near horizon limit

In the near horizon limit, i.e. small $z - z_*$, we define new coordinates ζ and τ through

$$z - z_* = -\frac{R_2^2 \lambda z_*^2}{\zeta}, \quad t = \lambda^{-1} \tau, \quad \lambda \ll 1,\tag{144}$$

where $R_2 = 1/\sqrt{d(d-1)}$. Furthermore, we study low frequencies ω such that $\Omega := \lambda^{-1} \omega$ is held fixed in the $\lambda \rightarrow 0$ limit. Expanding in small λ , we have that

$$\sqrt{g^{zz}} \partial_z \log T_\alpha^\pm = \frac{R_2^{-1} \zeta \Omega}{q e_d + \zeta \Omega \mp (-)^\alpha R_2 z_* k} + \mathcal{O}(\lambda),\tag{145}$$

so that Eq. (139) becomes

$$\begin{aligned} \left(\zeta \partial_\zeta \pm R_2 m - \frac{\zeta \Omega}{q e_d + \Omega \pm (-)^\alpha R_2 z_* k} \right) (\zeta \partial_\zeta \mp R_2 m) \Phi_\alpha^\pm - (R_2^2 z_*^2 k^2 - (q e_d + \zeta \Omega)^2) \Phi_\alpha^\pm &= 0, \\ \left(\zeta \partial_\zeta \mp R_2 m - \frac{\zeta \Omega}{q e_d + \zeta \Omega \mp (-)^\alpha R_2 z_* k} \right) (\zeta \partial_\zeta \pm R_2 m) \bar{\Phi}_\alpha^\pm - (R_2^2 z_*^2 k^2 - (q e_d + \zeta \Omega)^2) \bar{\Phi}_\alpha^\pm &= 0. \end{aligned} \quad (146)$$

For the special case $\Omega = 0$, alternatively in the $\zeta \rightarrow 0$ limit, one has the solutions

$$\Phi_\alpha^\pm = a_\alpha^\pm \zeta^{-\nu_k} + b_\alpha^\pm \zeta^{\nu_k}, \quad \nu_k := \sqrt{R_2^2 (m^2 + z_*^2 k^2) - q^2 e_d^2}, \quad (147)$$

for some (not independent) integration constants a_α^\pm and b_α^\pm . Hence, in the AdS_2 region, ν_k plays the role of an effective mass. Note that in fact the Dirac equation (146) yields analytic solutions in terms of the Whittaker functions [48, 47].

C.2 Schrödinger equation and flow equations

The Schrödinger equation consistent with the Heisenberg equation is

$$\begin{aligned} -\kappa^2 \partial_z \Psi[\eta^+, \bar{\eta}^-] &= H \Psi[\eta^+, \bar{\eta}^-], \\ H &:= \hat{H} \left(\Phi_\alpha^+ \rightarrow \eta_\alpha^+, \bar{\Phi}_\alpha^+ \rightarrow \frac{\delta}{\delta \eta_\alpha^+}, \Phi_\alpha^- \rightarrow -\frac{\delta}{\delta \bar{\eta}_\alpha^-}, \bar{\Phi}_\alpha^- \rightarrow \bar{\eta}_\alpha^- \right), \\ \frac{H}{\kappa^2} &= \int_k \sqrt{g_{zz}} \left[(\kappa_d^2)^2 \frac{\delta}{\delta \eta_\alpha^+} T_\alpha^- \frac{\delta}{\delta \bar{\eta}_\alpha^-} + \bar{\eta}_\alpha^- T_\alpha^+ \eta_\alpha^+ + \kappa_d^2 m \eta_\alpha^+ \frac{\delta}{\delta \eta_\alpha^+} + \kappa_d^2 m \bar{\eta}_\alpha^- \frac{\delta}{\delta \bar{\eta}_\alpha^-} \right]. \end{aligned} \quad (148)$$

Let us consider an ansatz for the wave-functional of the form⁴⁷

$$\Psi[\eta^+, \bar{\eta}^-] := \exp \left[- \int_k \left(\bar{\eta}_\alpha^- F_\alpha \eta_\alpha^+ + \bar{J}_\alpha^+ \eta_\alpha^+ + \bar{\eta}_\alpha^- J_\alpha^- + C_\alpha \right) \right], \quad (149)$$

where coefficients $F_\alpha, J_\alpha^-, \bar{J}_\alpha^+$ depend on z . Then, the Schrödinger equation yields the equations (39) for the coefficients with solutions given by (41).

C.3 Extension to higher dimensions

Here we give a comment on the extension to other dimensions. The choice of the representation of the $(1+3)$ -dimensional Dirac matrices was made to make the Dirac operator diagonal. In order to keep our analysis above even in higher dimensions, we have to keep the diagonal structure of the projectors Π_α in the extension to higher dimensions. For explicit constructions, it is convenient to introduce the tensor notation

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow A \otimes B = \begin{pmatrix} aB & bB \\ cB & dB \end{pmatrix}. \quad (150)$$

Then, the $(1+3)$ -dimensional case is

$$\Gamma^{\hat{z}} = 1 \otimes \sigma_3, \quad \Gamma^{\hat{t}} = 1 \otimes i\sigma_1, \quad \Gamma^{\hat{1}} = -\sigma_3 \otimes \sigma_2, \quad \Gamma^{\hat{2}} = \sigma_1 \otimes \sigma_2. \quad (151)$$

In $(1+4)$ -dimensions, $\Gamma^{\hat{3}}$ can be chosen to be proportional to a product of all Dirac matrices in $(1+3)$ -dimension.

$$\Gamma^{\hat{z}} = 1 \otimes \sigma_3, \quad \Gamma^{\hat{t}} = 1 \otimes i\sigma_1, \quad \Gamma^{\hat{1}} = -\sigma_3 \otimes \sigma_2, \quad \Gamma^{\hat{2}} = \sigma_1 \otimes \sigma_2, \quad \Gamma^{\hat{3}} = \sigma_2 \otimes \sigma_2. \quad (152)$$

In $(1+5)$ -dimension, in order to keep the diagonal structure of the projectors Π_α , the first three matrices $\Gamma^{\hat{z}}, \Gamma^{\hat{t}}, \Gamma^{\hat{1}}$ should have the same structure as the lower dimensional cases. This is realized by multiplying the three by the unit matrix of size 2.

$$\begin{aligned} \Gamma^{\hat{z}} &= 1 \otimes \sigma_3 \otimes 1, \quad \Gamma^{\hat{t}} = 1 \otimes i\sigma_1 \otimes 1, \quad \Gamma^{\hat{1}} = -\sigma_3 \otimes \sigma_2 \otimes 1, \\ \Gamma^{\hat{2}} &= \sigma_1 \otimes \sigma_2 \otimes 1, \quad \Gamma^{\hat{3}} = \sigma_2 \otimes \sigma_2 \otimes \sigma_2, \quad \Gamma^{\hat{4}} = \sigma_2 \otimes \sigma_2 \otimes \sigma_3. \end{aligned} \quad (153)$$

⁴⁷More generally, one could consider a term of the form $\bar{\eta}_\alpha^- F_{\alpha\beta} \eta_\beta^+$. However, since the Hamiltonian is diagonal, it is consistent, for simplicity, to concentrate on flows for which the off-diagonal components are put to zero.

In this way, we can continue explicit constructions of the Dirac matrices. Thus, all the analyses above hold for higher dimensions.

D Derivation of path integral from the operator formalism

In this section, we derive a path integral representation of a transition amplitude from the anti-commutation relations and the Hamiltonian. Especially, the boundary actions, which are usually required classically to ensure a well-defined variational problem of a classical action, will be derived quantum mechanically.

Let us derive the path integral representation of the following transition amplitude

$$\langle \chi_+, \bar{\chi}_- | P e^{-\int_0^l dz \hat{H}} | \bar{\chi}_+, \chi_- \rangle, \quad (154)$$

with a radial Hamiltonian in a quadratic form.⁴⁸

$$\hat{H} := \int_k \sqrt{g_{zz}} [\bar{\chi}_+ h_{++} \chi_+ + \bar{\chi}_+ h_{+-} \chi_- + \bar{\chi}_- h_{-+} \chi_+ + \bar{\chi}_- h_{--} \chi_-]. \quad (155)$$

A path integral representation is obtained by dividing a transition amplitude into a product of infinitesimal transition amplitudes with inserting the completeness relation

$$\prod_{k=1}^{N-1} \left[\int d^4 \chi_k \right] T(N, N-1) C(N-1) T(N-1, N-2) \cdots T(2, 1) C(1) T(1, 0), \quad (156)$$

where $\epsilon := l/N$, $d^4 \chi_k := d\chi_+^{(k)} d\bar{\chi}_+^{(k)} d\chi_-^{(k)} d\bar{\chi}_-^{(k)}$, and (0) and (N) correspond to the initial time and the final time, respectively. $T(k+1, k)$ and $C(k)$ are the transition amplitude from (k) to (k+1) and an exponential of the completeness relation, respectively:

$$\begin{aligned} T(k+1, k) &:= \langle \chi_+^{(k+1)}, \bar{\chi}_-^{(k+1)} | T e^{-\epsilon \hat{H}} | \bar{\chi}_+^{(k)}, \chi_-^{(k)} \rangle, \\ C(k) &:= e^{\int_k (\bar{\chi}_+^{(k)} \chi_+^{(k)} + \bar{\chi}_-^{(k)} \chi_-^{(k)})}. \end{aligned} \quad (157)$$

The infinitesimal transition amplitude $T(k, k-1)$ becomes

$$\begin{aligned} T(k, k-1) &= e^{-\epsilon H(k; k-1) + \int_k (-\bar{\chi}_+^{(k-1)} \chi_+^{(k)} - \bar{\chi}_-^{(k)} \chi_-^{(k-1)})}, \\ H(k, k-1) &:= \int_k \sqrt{g_{zz}} [\bar{\chi}_+^{(k-1)} h_{+-} \chi_-^{(k-1)} + \bar{\chi}_-^{(k)} h_{-+} \chi_+^{(k)} + \bar{\chi}_+^{(k)} h_{++} \chi_+^{(k-1)} - \bar{\chi}_-^{(k)} h_{--} \chi_-^{(k-1)}]. \end{aligned}$$

Let us see the exponents of $T(k+1, k)C(k)T(k, k-1)C(k-1)$ in (156),

$$\begin{aligned} & -\epsilon H(k+1, k) - \epsilon H(k, k-1) \\ & + \int_k \left[\alpha(-\bar{\chi}_+^{(k)} \chi_+^{(k+1)} - \bar{\chi}_-^{(k+1)} \chi_-^{(k)}) + \beta(-\bar{\chi}_+^{(k)} \chi_+^{(k+1)} - \bar{\chi}_-^{(k+1)} \chi_-^{(k)}) \right. \\ & \quad + \alpha(\bar{\chi}_+^{(k)} \chi_+^{(k)} + \bar{\chi}_-^{(k)} \chi_-^{(k)}) + \beta(\bar{\chi}_+^{(k)} \chi_+^{(k)} + \bar{\chi}_-^{(k)} \chi_-^{(k)}) \\ & \quad + \alpha(-\bar{\chi}_+^{(k-1)} \chi_+^{(k)} - \bar{\chi}_-^{(k)} \chi_-^{(k-1)}) + \beta(-\bar{\chi}_+^{(k-1)} \chi_+^{(k)} - \bar{\chi}_-^{(k)} \chi_-^{(k-1)}) \\ & \quad \left. + \alpha(\bar{\chi}_+^{(k-1)} \chi_+^{(k-1)} + \bar{\chi}_-^{(k-1)} \chi_-^{(k-1)}) + \beta(\bar{\chi}_+^{(k-1)} \chi_+^{(k-1)} + \bar{\chi}_-^{(k-1)} \chi_-^{(k-1)}) \right], \end{aligned} \quad (158)$$

where each exponent of T, C is divided into two parts through $\alpha + \beta = 1$. First, let us extract four terms from the terms with α -coefficients, which are the first term in the second line, the first and second terms in the third line and the second term in the fourth line, of the last expression (158). The sum of the four terms becomes

$$-\alpha \sum_{k=1}^{N-1} \bar{\chi}_+^{(k)} (\chi_+^{(k+1)} - \chi_+^{(k)}) + \alpha \sum_{k=0}^{N-2} \bar{\chi}_-^{(k+1)} (\chi_-^{(k+1)} - \chi_-^{(k)}). \quad (159)$$

⁴⁸ This is just for simplicity. Actually, we do not have to limit ourselves to a quadratic form as seen from the following derivation.

One can see that this becomes a kinetic term after the $\epsilon \rightarrow 0$ limit. Similarly, let us extract four terms from the terms with β -coefficients, which are the fourth term in the second line, the third and fourth terms in the third line and the third term in the fourth line, of (158). The sum of the four terms becomes

$$\beta \sum_{k=0}^{N-2} (\bar{\chi}_+^{(k+1)} - \bar{\chi}_+^{(k)}) \chi_+^{(k+1)} - \beta \sum_{k=1}^{N-1} (\bar{\chi}_-^{(k+1)} - \bar{\chi}_-^{(k)}) \chi_+^{(k)}. \quad (160)$$

One can see that this becomes a kinetic term after the $\epsilon \rightarrow 0$ limit. Then, gathering all factors and extracting the exponent taking these two types of combinations into account, we find

$$\begin{aligned} & -\epsilon \sum_{k=0}^{N-1} H(k+1, k) \\ & + \int_k \epsilon \left[-\alpha \sum_{k=1}^{N-1} \bar{\chi}_+^{(k)} (\chi_+^{(k+1)} - \chi_+^{(k)}) \epsilon^{-1} + \alpha \sum_{k=0}^{N-2} \bar{\chi}_-^{(k+1)} (\chi_-^{(k+1)} - \chi_-^{(k)}) \epsilon^{-1} \right. \\ & \quad \left. + \beta \sum_{k=0}^{N-2} (\bar{\chi}_+^{(k+1)} - \bar{\chi}_+^{(k)}) \epsilon^{-1} \chi_+^{(k+1)} - \beta \sum_{k=1}^{N-1} (\bar{\chi}_-^{(k+1)} - \bar{\chi}_-^{(k)}) \epsilon^{-1} \chi_-^{(k)} \right] \\ & + \int_k \left[-\alpha \bar{\chi}_-^{(N)} \chi_-^{(N-1)} - \beta \bar{\chi}_+^{(N-1)} \chi_+^{(N)} - \beta \bar{\chi}_-^{(1)} \chi_-^{(0)} - \alpha \bar{\chi}_+^{(0)} \chi_+^{(1)} \right]. \end{aligned} \quad (161)$$

The second line results from the first combination (159) and the third line results from the second combination (160). After taking $N \rightarrow \infty$, we find

$$\begin{aligned} & \int dz \left[\int_k (-\alpha \bar{\chi}_+ \partial_z \chi_+ + \alpha \bar{\chi}_- \partial_z \chi_- + \beta \partial_z \bar{\chi}_+ \chi_+ - \beta \partial_z \bar{\chi}_- \chi_-) - H \right] \\ & + \int_k [-\alpha \bar{\chi}_- \chi_- - \beta \bar{\chi}_+ \chi_+]_{z=l} + \int_k [-\alpha \bar{\chi}_+ \chi_+ - \beta \bar{\chi}_- \chi_-]_{z=\epsilon}, \end{aligned} \quad (162)$$

which consists of $-S$ plus boundary actions at two ends. The boundary actions coincide with those obtained [43]. One can see that partial integrations of the kinetic terms of the Dirac action corresponds to changes of pairings in taking all products of the infinitesimal transition amplitudes.

E Counterterm action

In AdS/CFT, if we start from the classical action with no counterterm action, it is difficult to take a well-defined continuum limit $\epsilon_0 \rightarrow 0$ and get a correct answer. For example, a two-point function generally contains contact terms with integer powers of momentum k_μ , which dominate over non-local terms with non-integer powers of k_μ in the continuum limit. Since the non-local term with the lowest power gives the correct scaling behavior of two-point functions, we need to eliminate the dominating contact terms. The counterterm action is added to fulfil this purpose.⁴⁹ We adopt the Poincare coordinate $ds^2 \sim z^{-2}(dz^2 + \dots)$ near the AdS boundary. For details of holographic renormalization and the counterterm action, see, *e.g.*, [16, 58].

E.1 Scalar case

The GKP-Witten relation with the counterterm action in our language is

$$\left\langle \exp \int_k \phi_0 O \right\rangle_{\text{ct}, \epsilon_0}^{\text{std}} = \langle \text{IR} | \hat{U}(\epsilon_{\text{IR}}, \epsilon_0) e^{\hat{S}_{\text{ct}}^0} | \phi_0 \rangle. \quad (163)$$

⁴⁹ Of course, in addition, the rescaling of boundary fields is also needed to make correlation functions finite in the $\epsilon_0 \rightarrow 0$ limit.

Here ϵ_0 is a UV cutoff and S_{ct} is the counterterm action. One choice of the counterterm action is⁵⁰

$$\widehat{S}_{\text{ct}}^0 := \frac{1}{2} \int_k A_0 \widehat{\phi}^2, \quad A_0 := \frac{\pi_-(\epsilon_0)}{\phi_-(\epsilon_0)} = -\epsilon_0^{-d} (\Delta_- + O((k\epsilon_0)^2) + \dots), \quad (164)$$

where we introduced solutions ϕ_{\pm} to the classical equation of motion and their conjugate momenta π_{\pm} ,

$$\phi_{\pm}(z) := c_{\pm} z^{d/2} I_{\pm\nu}(kz) = z^{\Delta_{\pm}} + \dots, \quad \pi_{\pm}(z) := -\sqrt{g} g^{zz} \partial_z \phi_{\pm}(z). \quad (165)$$

Let us consider the following question: if a solution to the radial Schrödinger equation is given, what couplings does the UV generating functional contain? A general solution to the radial Schrödinger equation can be written as

$$\langle \phi | \widehat{U}(\epsilon, \epsilon_0) e^{\widehat{S}_{\text{ct}}^0} | \Psi_0 \rangle = \exp \left[\int_k \left(-\frac{1}{2} f(\epsilon) \phi^2 + J(\epsilon) \phi \right) + C(\epsilon) \right]. \quad (166)$$

Then, the above question is to find a UV generating function corresponding to $|\Psi_0\rangle$, which are derived as follows:

$$\begin{aligned} \langle \text{IR} | \widehat{U}(\epsilon_{\text{IR}}, \epsilon_0) e^{\widehat{S}_{\text{ct}}^0} | \Psi_0 \rangle &= \int D\phi_0 \langle \text{IR} | \widehat{U}(\epsilon_{\text{IR}}, \epsilon_0) e^{\widehat{S}_{\text{ct}}^0} | \phi_0 \rangle \langle \phi_0 | \Psi_0 \rangle \\ &= \left\langle \exp \left(\frac{1}{2} \int_k \frac{O^2}{f(\epsilon_0) + A_0} + \int_k \frac{J(\epsilon_0) O}{f(\epsilon_0) + A_0} + \dots \right) \right\rangle_{\text{ct}, \epsilon_0}^{\text{std}}. \end{aligned} \quad (167)$$

E.2 Fermion case

The discussion is parallel to the scalar case. The GKP-Witten relation with the counterterm action is

$$\left\langle \exp \left(\int_k [\overline{\chi}_+^0 O_+ + \overline{O}_- \chi_-^0] \right) \right\rangle_{\text{ct}, \epsilon_0}^{\text{std}} = \langle \text{IR} | \widehat{U}(\epsilon_{\text{IR}}, \epsilon_0) e^{\widehat{S}_{\text{ct}}^0} | \overline{\chi}_+^0, \chi_-^0 \rangle. \quad (168)$$

One choice of the counterterm action is

$$\widehat{S}_{\text{ct}}^0 := \int_k \widehat{\chi}_+ A_0 \widehat{\chi}_-, \quad A_0 := \frac{\phi_+(\epsilon_0)}{\phi_-(\epsilon_0)}, \quad (169)$$

where we used the same notation as (140). Note that $\phi_- \sim z^{-m}$ if a fermion is of mass m . Now, let us assume that we have a solution to the radial Schrödinger equation given by

$$\langle \chi_+, \overline{\chi}_- | \widehat{U}(\epsilon, \epsilon_0) e^{\widehat{S}_{\text{ct}}^0} | \Psi_0 \rangle = \exp \left(- \int_k [\overline{\chi}_- F(\epsilon) \chi_+ + \overline{J}_+(\epsilon) \chi_+ + \overline{\chi}_- J_-(\epsilon)] \right). \quad (170)$$

Then, the corresponding UV generating functional is

$$\begin{aligned} &\langle \text{IR} | \widehat{U}(\epsilon_{\text{IR}}, \epsilon_0) e^{\widehat{S}_{\text{ct}}^0} | \Psi_0 \rangle \\ &= \left\langle \exp \int_k \left[-\overline{O}_- \frac{1}{F(\epsilon_0)^{-1} - A_0} O_+ + \overline{J}_+(\epsilon_0) \frac{F(\epsilon_0)^{-1}}{F(\epsilon_0)^{-1} - A_0} O_+ + \overline{O}_- \frac{F(\epsilon_0)^{-1}}{F(\epsilon_0)^{-1} - A} J_-(\epsilon_0) \right] \right\rangle_{\text{ct}, \epsilon_0}^{\text{std}}. \end{aligned} \quad (171)$$

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⁵⁰ In general, A_0 can take different forms: for example, a minimal choice is to take terms which have smaller (integer) powers than the smallest non-integer power 2ν in the expansion of $\epsilon_0^d A_0$ in $k\epsilon_0$. The remaining integer powers become subleading in $\epsilon_0 \rightarrow 0$ limit since they are larger than 2ν . This remark holds in the fermion case, as well.

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